Assignment #3: Linear Independence, Linear Transformations and Linear Models, Applications

Due date: Monday, February 16, 2015 (4:00pm)

Name: ________________________________________________________________

Section Number
1) Are the columns of the following matrices linearly independent? Explain how you know, in each case. Don’t use Matlab for this question. You should be able to answer all three by inspection.

\[ \begin{bmatrix} 3 & -1 & 12 \\ 7 & 4 & 1 \end{bmatrix} \quad \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & 2 & 3 & 5 \\ 2 & 3 & 5 & 8 \\ 3 & 4 & 7 & 11 \\ 4 & 5 & 9 & 14 \end{bmatrix} \]

2) For what values of \( h \) will the vectors in each of the following sets be linearly independent? Briefly explain your answer in each case.

\[ \begin{bmatrix} 0 & h & 1 \\ 1 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 1 \end{bmatrix} \quad \begin{bmatrix} 2 & -1 & h \\ 1 & 1 & 0 \end{bmatrix} \quad \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 2 & h \end{bmatrix} \]

3) For each of the following statements, assume that the first clause is true and indicate whether the second clause will be always, sometimes or never true. Please justify your answers.

a) \( \{v_1, \ldots, v_m\} \) is a set of linearly independent vectors in \( \mathbb{R}^n \); \( n \geq m \).

b) A homogeneous system \( Ax = 0 \) has the trivial solution; the columns of \( A \) are linearly independent.

c) Three vectors \( v_1, v_2 \) and \( v_3 \) are in \( \mathbb{R}^3 \) and \( v_3 \) is not a linear combination of \( v_1 \) and \( v_2 \); the set \( \{v_1, v_2, v_3\} \) is linearly independent.

4) For each of the following, give an example or explain why it is impossible to construct one:

a) A \( 3 \times 2 \) coefficient matrix whose columns span \( \mathbb{R}^2 \)

b) A \( 2 \times 3 \) coefficient matrix whose columns span \( \mathbb{R}^2 \)

c) A \( 3 \times 2 \) coefficient matrix whose columns span \( \mathbb{R}^3 \)

d) A \( 3 \times 3 \) coefficient matrix whose columns span \( \mathbb{R}^3 \)

5) Let \( A = \begin{bmatrix} 1 & 2 & 0 \\ -2 & -3 & -1 \\ 1 & 1 & 1 \end{bmatrix} \) and observe that the first column is equal to half of the sum of the second and third columns. Without reducing the matrix, find a non-trivial solution to the homogeneous equation \( Ax = 0 \).
6) Give a geometric description of each of the following transformations \( x \rightarrow Ax \):

a) \( A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \) 

b) \( A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \) 

c) \( A = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \)

7) If \( A \) is a \( 3 \times 4 \) matrix that represents a linear transformation \( T \), what is the domain of \( T \)? What is the co-domain of \( T \)?

8) If \( A = \begin{bmatrix} 1 & 0 & 3 \\ 0 & 2 & 1 \\ 4 & 1 & 0 \end{bmatrix} \) and \( x = \begin{bmatrix} -1 \\ 3 \\ 2 \end{bmatrix} \), what is the image of \( x \) under the transformation \( T(x) = Ax \)?

9) If \( A = \begin{bmatrix} 2 & 3 & 1 \\ 1 & 0 & -1 \\ 3 & 1 & 2 \end{bmatrix} \), what is the range of the transformation \( T(x) = Ax \)?

10) Find all \( x \in \mathbb{R}^4 \) that are mapped onto the zero vector by the transformation \( x \rightarrow Ax \) where

\[
A = \begin{bmatrix} 2 & -3 & 4 & 3 \\ 1 & 6 & 1 & 4 \\ 3 & 3 & 1 & 7 \end{bmatrix}
\]

11) If \( T \) is a linear transformation from \( \mathbb{R}^2 \) to \( \mathbb{R}^3 \) and \( T(e_1) = (1, 0, -2) \) and \( T(e_2) = (-1, 0, 2) \) where \( e_1 \) and \( e_2 \) are the columns of the \( 2 \times 2 \) identity matrix, what is the standard matrix of the transformation \( T \)? Which of the following statements best describe what this transformation does? Please justify your answer.

- a) Project points from 2D space into a 3D volume
- b) Maps a set of vectors that span a 2D plane onto a set of vectors that span a line
- c) Both a) and b)
- d) Neither a) nor b)

12) What is the standard matrix of the linear transformation \( T: \mathbb{R}^2 \rightarrow \mathbb{R}^2 \) that first rotates points, in a counterclockwise direction, by 90° about the origin and then reflects them through the line \( x = y \)?

13) Let \( T: \mathbb{R}^2 \rightarrow \mathbb{R}^2 \) be a linear transformation with the standard matrix \( A = \begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix} \). Sketch the images of the vectors \( \begin{bmatrix} 0 \\ 1 \end{bmatrix} \), \( \begin{bmatrix} 1 \\ 0 \end{bmatrix} \), \( \begin{bmatrix} 1 \\ 1 \end{bmatrix} \) under the transformation \( T \).
14) For each of the following, choose the appropriate answer:
   a) If $T: \mathbb{R}^n \to \mathbb{R}^m$ and $n > m$
      i) $T$ will {always / sometimes / never} map $\mathbb{R}^n$ onto $\mathbb{R}^m$
      ii) $T$ will {always / sometimes / never} be one-to-one
   b) If $T: \mathbb{R}^n \to \mathbb{R}^m$ and $n = m$
      i) $T$ will {always / sometimes / never} map $\mathbb{R}^n$ onto $\mathbb{R}^m$
      ii) $T$ will {always / sometimes / never} be one-to-one
   c) If $T: \mathbb{R}^n \to \mathbb{R}^m$ and $n < m$
      i) $T$ will {always / sometimes / never} map $\mathbb{R}^n$ onto $\mathbb{R}^m$
      ii) $T$ will {always / sometimes / never} be one-to-one

15) The Solvay process produces sodium carbonate (soda ash) from brine and limestone. The overall process is shown below:

   \[ 2\text{NaCl} + \text{CaCO}_3 \to \text{Na}_2\text{CO}_3 + \text{CaCl}_2 \]

   For each compound, construct a vector that lists the numbers of atoms of sodium (Na), chlorine (Cl), calcium (Ca), carbon (C) and oxygen (O), and then show how the methods taught in this class can be used to determine the smallest integer weights that can be applied to each compound to balance the chemical equations.

16) Suppose that you are designing a menu of specialty coffees and you would like to offer customized combinations of calories, fat, and protein to each client by varying the amounts of cream, sugar, and milk used in your recipe. Suppose that:

   - 4 oz. of cream provides 80 calories, 100 grams of fat, and 30 grams of protein
   - 4 oz. of sugar provides 70 calories, 20 grams of fat, and 10 grams of protein
   - 4 oz. of milk provides 50 calories, 25 grams of fat, and 100 grams of protein

   a) If Kelly wants to order a drink that contains 410 calories, 370 grams of fat, and 300 grams of protein, how many ounces of cream, sugar, and milk should be added?

   b) Can you use the same approach to determine the optimal amounts of cream, sugar, and milk for a client if the targets are: 240 calories, 255 fat, 350 protein? Why or why not?

17) Suppose that the population distribution in the Minneapolis area is defined by a recurrence relation $x_{t+1} = Ax_t$ for $t = 0, 1, \ldots$, so that, for example, the population in the year 2015 is defined as a function of the population in year 2014. Suppose that the Minneapolis area has a population of 1 million people in 2014, 60% of whom live in the city and 40% of whom live in the suburbs, and that each year 5% of the population in the city will move to the suburbs (while 95% stays in the city) and 3% of the population in the suburbs will move to the city (while 97% stays in the suburbs). Please note that this is an extremely simplified model in which the total population remains stable and all that changes is how people are distributed between the city and the suburbs: no new people move to the Minneapolis area, no one is born and no one dies.

   a) What is the system of linear equations that you can use to predict the size of the population in the city and in the suburbs in 2015 as a function of the size of the population in the city and in the suburbs in 2014?

   b) Use one iteration of this linear model to predict the population of the city and suburbs in 2015.

   c) What do you intuitively predict will happen to the population in the limit? Use Matlab to carry out 1000 iterations of the model. What do you find?