Assignment #7: Determinants, Eigenvalues, and Eigenvectors

Due date: Monday, April 6, 2015 (4:00pm)

Name: ____________________________________________________________

Section Number
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For full credit you must show all of your work.

1. Compute
\[
\begin{vmatrix}
1 & 0 & 2 & 0 \\
0 & 2 & 1 & -1 \\
2 & 1 & 0 & 2 \\
-1 & 0 & 1 & 1
\end{vmatrix}
\]
using co-factor expansion. You can check your answer using Matlab.

(5 pts)

2. Compute
\[
\begin{vmatrix}
1 & -1 & 0 & 1 \\
0 & 1 & -1 & 0 \\
2 & -2 & 1 & 0 \\
-1 & 1 & 1 & 2
\end{vmatrix}
\]
using row reduction. You can check your answer using Matlab.

(5 pts)

3. Compute
\[
\begin{vmatrix}
1 & 2 & 2 & 0 \\
0 & 1 & 0 & -1 \\
2 & -1 & 4 & 2 \\
-1 & 0 & -2 & 1
\end{vmatrix}
\]
by inspection. Explain your answer. (5 pts)

4. Compute
\[
\begin{vmatrix}
1 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 \\
2 & 0 & 2 & 0 \\
-1 & 0 & -2 & 1
\end{vmatrix}
\]
by inspection. Explain your answer. (5 pts)

5. For an arbitrary matrix
\[
A = \begin{bmatrix}
a & b \\
c & d
\end{bmatrix}
\]
and scalar \(k\), find a formula that relates \(\det(kA)\) to \(\det(A)\).

(5 pts)

6. Find a formula that relates \(\det(kA)\) to \(\det(A)\), for any arbitrary \(n \times n\) matrix \(A\) and scalar \(k\). (5 pts)

7. Write a series of short programs in Matlab to explore the following questions: (5 pts each)
   a. Is it true that \(\det(A + B) = \det(A) + \det(B)\), for general \(n \times n\) matrices \(A\) and \(B\)?
      [In a loop bounded by a small fixed number of iterations, generate random \(n \times n\) matrices \(A\) and \(B\), and compute \textit{error} = \(\det(A + B) - (\det(A) + \det(B))\). Report what you find, and provide a printout showing your work.]
   b. Is it true that \(\det(AB) = \det(A) \cdot \det(B)\), for general \(n \times n\) matrices \(A\) and \(B\)? [Follow a similar approach as in part a above.]
   c. Compare \(\det(A)\) with each of the following: \(\det(A^T)\), \(\det(-A)\) and \(\det(A^{-1})\) for simple square matrices of various sizes and make conjectures about how each of these determinants are
related. Provide a printout showing your work. You may find it helpful to use the \texttt{format}
command in Matlab to output some of your results in rational form.

d. Compute $A^TA$, $AA^T$, $\det(A^TA)$ and $\det(AA^T)$ for several random $m \times n$ matrices where $m > n$. What do you observe?

8. Try to solve the following by eye. You can check your answers using Matlab. (2 pts each)

If $\begin{pmatrix} 0 & 6 & 8 \\ 1 & 2 & 4 \\ 2 & 1 & 3 \end{pmatrix} = 6$, what is:

\[
\begin{array}{ccc}
\text{a.} & \begin{pmatrix} 0 & -3 & -4 \\ 1 & 2 & 4 \\ 2 & 1 & 3 \end{pmatrix} & \text{b.} & \begin{pmatrix} 0 & 6 & 8 \\ 1 & 2 & 4 \\ 1 & -1 & -1 \end{pmatrix} & \text{c.} & \begin{pmatrix} 2 & 1 & 3 \\ 0 & 6 & 8 \\ 1 & 2 & 4 \end{pmatrix}
\end{array}
\]

9. Two square matrices $A$ and $B$ are said to be \textit{similar} if there exists a matrix $M$ such that $B = MA M^{-1}$. Show that similar matrices have the same determinants. (5 pts)

10. Is $\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$ an eigenvector of $A = \begin{pmatrix} 5 & 2 & 3 \\ 3 & 8 & 3 \\ 2 & 0 & 4 \end{pmatrix}$? Justify your answer. (3 pts)

11. Is 5 an eigenvalue of $A = \begin{pmatrix} 5 & 2 & 3 \\ 3 & 8 & 3 \\ 2 & 0 & 4 \end{pmatrix}$? Justify your answer. (6 pts)

12. What are the eigenvalues and eigenvectors of $A = \begin{pmatrix} 0 & -1 & -1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}$? (12 pts)

13. For each of the following statements, indicate whether it is \textbf{True} or \textbf{False}: (2 pts each)

a. If $\mathbf{v}_1$ and $\mathbf{v}_2$ are linearly independent eigenvectors of a matrix $A$, then they correspond to distinct eigenvalues of $A$.

b. The eigenspace of a matrix $A$ is the union of the null spaces of each of the matrices $(A - \lambda_i I)$ where $\lambda_i$ are the eigenvalues of $A$.

c. An $n \times n$ matrix can have at most $n$ real eigenvalues

d. If $\lambda$ is a non-zero eigenvalue of an invertible matrix $A$ then $1/\lambda$ is an eigenvalue of $A^{-1}$.

e. For any square matrix $A$, $A$ and $A^T$ have the same eigenvalues.
f. For any square matrix $A$, $A$ and $A^T$ have the same eigenvectors.

g. If two matrices $A$ and $B$ are similar, then they will have the same eigenvalues.

h. If two matrices $A$ and $B$ have the same eigenvalues, then they are similar.

i. If a matrix $B$ is obtained from a matrix $A$ using row replacement operations only (no scaling, no row swapping), then $A$ and $B$ will have the same determinant and hence the same eigenvalues.