Assignment #2: Vector and Matrix Equations, Solution Sets of Linear Systems

Due date: Friday, February 12, 2016 (10:10am)

Name: ________________________________________________________________

Section Number
Assignment #2: Vector and Matrix Equations, Solution Sets of Linear Systems

Due date: Friday, February 12, 2016 (10:10am)
For full credit you must show all of your work.

Some of the questions below are derived from examples in Lay, Linear Algebra and its Applications.

1) Using the function `make_echelon.m` as a guide, write a function in Matlab that converts a matrix into echelon form. Test your program on the following input:

\[
\begin{bmatrix}
2 & 2 & 9 & 7 \\
1 & 0 & -3 & 8 \\
0 & 1 & 5 & -2
\end{bmatrix}
\begin{bmatrix}
0 & 2 & 3 & 3 \\
2 & 3 & 1 & 5 \\
1 & 0 & 2 & 4
\end{bmatrix}
\begin{bmatrix}
3 & -3 & 1 & 0 \\
-2 & 2 & 5 & 0 \\
1 & -1 & -2 & 0
\end{bmatrix}
\begin{bmatrix}
1 & -4 & 7 & 2 \\
0 & 3 & -5 & 1 \\
-2 & 5 & -9 & 0
\end{bmatrix}
\]

```matlab
function [ b ] = make_echelon( a )
    %this function accepts an input matrix a and produces as output a matrix b
    %that has the same solution as a but is in echelon form
    nrows = size(a,1);
    ncols = size(a,2);
    % initialize b = a
    b = a;
    % start at the first row, first column
    firstrow = 1;
    for j = 1:ncols
        % <if it is a zero column, skip on>
        % <if there is a zero in the pivot position, swap rows to obtain a non-zero pivot>
        for i = (firstrow+1):1:nrows;
            % <use row replacement operations to eliminate the elements in b that are below the pivot>
            firstrow = firstrow + 1;
        end
    end
end
```

2) Express this system as a vector equation and as a matrix equation:

\[3x + 4y + z = 1 \]
\[y - 4w = 2 \]
\[2y + 6z + 4w = -1 \]
\[x - 5y + 5z - 3w = 0 \]

2) Express this system as a vector equation and as a matrix equation:

\[
\begin{bmatrix}
1 & 3 & 5 & 4 \\
1 & 2 & 4 & 3 \\
1 & 2 & k & l
\end{bmatrix}
\]

3) Given the augmented matrix representing a system of linear equations,

a) Reduce this system to echelon form

b) Provide values for k and l such that the system has:
   i) one unique solution       ii) an infinite number       iii) no solution
   of solutions
c) If an equation of the form $ax + by + cz = d$ represents a plane in $\mathbb{R}^3$, then each row of the above augmented system can be considered as representing a plane. Explain the geometric situation when the system has:

i) one unique solution 
ii) an infinite number 
iii) no solutions 

of solutions

4) Fill in the blanks to make the following system consistent: (hint: you do not need to reduce)

\[
\begin{bmatrix}
-2 & -6 & 1 \\
-1 & -11 & 2 \\
-2 & 1 & -1
\end{bmatrix} \begin{bmatrix}
x \\
y \\
z
\end{bmatrix} = \begin{bmatrix}
4 \\
-1 \\
-
\end{bmatrix}
\]

5) You may use Matlab to find the answer to this question, but please explain your reasoning.

Let $v_1 = \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}$, $v_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$, $v_3 = \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix}$. Also, let $u = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$ and $w = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

a) Is $u$ in the subset of $\mathbb{R}^3$ spanned by $\{v_1, v_2, v_3\}$? If so, what weights can be applied to each of $v_1, v_2, v_3$ to produce $u$?

b) Is $w$ in the subset of $\mathbb{R}^3$ spanned by $\{v_1, v_2, v_3\}$? If so, what weights can be applied to each of $v_1, v_2, v_3$ to produce $w$?

6) Let $v_1 = \begin{bmatrix} 3 \\ -1 \\ 5 \end{bmatrix}$, $v_2 = \begin{bmatrix} 2 \\ 5 \\ -2 \end{bmatrix}$, $v_3 = \begin{bmatrix} 1 \\ -1/3 \\ 5/3 \end{bmatrix}$, $v_4 = \begin{bmatrix} -1 \\ 2 \\ -1 \end{bmatrix}$, $v_5 = \begin{bmatrix} 2 \\ 0 \\ 3 \end{bmatrix}$.

a) Provide a geometric description of:

i) span$\{v_1, v_3\}$

ii) span$\{v_1, v_2\}$

iii) span$\{v_1, v_2, v_4, v_3\}$

iv) span$\{v_1, v_2, v_4\}$

b) Is $v_4$ in span$\{v_1, v_2, v_4\}$?

c) Is $v_3$ in span$\{v_1, v_2\}$?

7) Let $v_1 = \begin{bmatrix} 3 \\ -1 \\ -3 \\ 1 \end{bmatrix}$, $v_2 = \begin{bmatrix} 1 \\ -3 \\ -1 \\ 3 \end{bmatrix}$, and $v_3 = \begin{bmatrix} 6 \\ -6 \\ -6 \\ 6 \end{bmatrix}$. Without doing any computation, answer the following questions:

a) Do the three vectors $\{v_1, v_2, v_3\}$ span $\mathbb{R}^4$? Why or why not?

b) Do the three vectors $\{v_1, v_2, v_3\}$ span $\mathbb{R}^3$? Why or why not?
8) Note that \[
\begin{bmatrix}
-8 & 5 & 3 \\
1 & -1 & 3 \\
2 & -7 & 2
\end{bmatrix}
\begin{bmatrix}
4 \\
2 \\
3
\end{bmatrix} =
\begin{bmatrix}
-13 \\
11 \\
0
\end{bmatrix}.
\]
Use this fact (and only this fact) to find scalars \(c_1\), \(c_2\), and \(c_3\) such that:
\[
\begin{bmatrix}
-13 \\
11 \\
0
\end{bmatrix} =
\begin{bmatrix}
-8 \\
5 \\
-7
\end{bmatrix}c_1 +
\begin{bmatrix}
1 \\
-1 \\
2
\end{bmatrix}c_2 +
\begin{bmatrix}
3 \\
3 \\
2
\end{bmatrix}c_3.
\]

9) Suppose that \[
\begin{bmatrix}
1 & 0 & 2 & 4 \\
3 & 1 & 0 & -2 \\
-1 & 0 & -3 & 1
\end{bmatrix}
\begin{bmatrix}
1 \\
3 \\
-1
\end{bmatrix} =
\begin{bmatrix}
5 \\
4 \\
0
\end{bmatrix}
\]
and \[
\begin{bmatrix}
1 & 0 & 2 & 4 \\
3 & 1 & 0 & -2 \\
-1 & 0 & -3 & 1
\end{bmatrix}
\begin{bmatrix}
-2 \\
2 \\
1
\end{bmatrix} =
\begin{bmatrix}
-8 \\
1 \\
1
\end{bmatrix}.
\]
Use this fact, and only this fact, to solve for \(x\) in the equation:
\[
\begin{bmatrix}
1 & 0 & 2 & 4 \\
3 & 1 & 0 & -2 \\
-1 & 0 & -3 & 1
\end{bmatrix}
\begin{bmatrix}
10 \\
3 \\
-1
\end{bmatrix} =
\begin{bmatrix}
13 \\
-4 \\
1
\end{bmatrix}.
\]
Explain your answer. (Hint: note that \[
\begin{bmatrix}
13 \\
-4 \\
1
\end{bmatrix} =
\begin{bmatrix}
5 \\
4 \\
0
\end{bmatrix} +
\begin{bmatrix}
8 \\
-8 \\
1
\end{bmatrix}.
\]

10) Suppose \(A\) is a 4x3 matrix and \(b\) is a vector in \(\mathbb{R}^4\) with the property that \(Ax = b\) has a unique solution. What is the reduced echelon form of \(A\)? How do you know?

11) \[
\begin{bmatrix}
1 & -2 & 2 \\
-1 & 0 & 2 \\
0 & 1 & -2
\end{bmatrix}
\begin{bmatrix}
1 \\
2 \\
3
\end{bmatrix} =
\begin{bmatrix}
3 \\
5 \\
-4
\end{bmatrix}.
\]
Are there any other vectors \(x\) for which \[
\begin{bmatrix}
1 & -2 & 2 \\
-1 & 0 & 2 \\
0 & 1 & -2
\end{bmatrix}
\begin{bmatrix}
1 \\
0 \\
0
\end{bmatrix} =
\begin{bmatrix}
3 \\
5 \\
-4
\end{bmatrix}?
\]
If so, give one example. Describe the solution set of \[
\begin{bmatrix}
1 & -2 & 2 \\
-1 & 0 & 2 \\
0 & 1 & -2
\end{bmatrix}
\begin{bmatrix}
1 \\
-1 \\
0
\end{bmatrix} =
\begin{bmatrix}
3 \\
5 \\
-4
\end{bmatrix}.
\]

12) 
\(a\) Give an example of a 3x3 matrix \(A\) that has no zero elements and for which \(Ax = b\) has a solution for all possible \(b\).
\(b\) Give an example of a 3x3 matrix \(A\) that has no zero elements and for which \(Ax = b\) does not have a solution for all possible \(b\) (i.e. for which there exist some vectors \(b\) that cannot be achieved as a result of multiplying \(A\) by any vector \(x\)).
\(c\) Let \(A\) be a 3x2 matrix. Is there any \(A\) that is consistent for every \(b\) in \(\mathbb{R}^3\)? Explain.
13) Express the solution set of the following system in parametric vector form:

\[
\begin{align*}
  x_1 - x_2 + 3x_3 &= 0 \\
  -x_1 + 2x_2 - 3x_3 &= 0 \\
  -2x_1 - 6x_3 &= 0
\end{align*}
\]

14) Describe and compare the solution sets of the following two systems:

\[
\begin{align*}
  x_1 + x_2 - x_3 &= 0 \\
  x_1 + x_2 - x_3 &= -10
\end{align*}
\]

15) Matlab allows us to plot the plane spanned by two vectors in 3D using the following syntax:

\[
\begin{align*}
  v1 &= [v1x; v1y; v1z]; \\
  v2 &= [v2x; v2y; v2z]; \\
  fx &= @(s,t) v1(1)*s + v2(1)*t; \\
  fy &= @(s,t) v1(2)*s + v2(2)*t; \\
  fz &= @(s,t) v1(3)*s + v2(3)*t; \\
  ezsurf(fx, fy, fz);
\end{align*}
\]

Note that \((v1x, v1y, v1z)\) are the elements of the vector \(v1\), and \((v2x, v2y, v2z)\) are the elements of the vector \(v2\). Therefore, these are values you need to specify. The remaining lines of code plot the surface that is spanned by these two vectors.

To displace the plane by an offset vector \(p = [px; py; pz]\), one can use the following:

\[
\begin{align*}
  fx &= @(s,t) p(1) + v1(1)*s + v2(1)*t; \\
  fy &= @(s,t) p(2) + v1(2)*s + v2(2)*t; \\
  fz &= @(s,t) p(3) + v1(3)*s + v2(3)*t;
\end{align*}
\]

Note again that \(p\) is a vector that you need to define, in order to specify the offset.

Use the above approach, along with your answers to question 14, to visualize the solution sets of the two systems: \(x_1 + x_2 - x_3 = 0\) and \(x_1 + x_2 - x_3 = -10\). Please turn in a screen shot of the result.

16) For 10% extra credit: Write a function in Matlab that converts a matrix into reduced echelon form. Test your program on the following input:

\[
\begin{bmatrix}
  2 & 2 & 9 & 7 \\
  1 & 0 & -3 & 8 \\
  0 & 1 & 5 & -2
\end{bmatrix}
\]

\[
\begin{bmatrix}
  0 & 2 & 3 & 3 \\
  2 & 3 & 1 & 5 \\
  1 & 0 & 2 & 4
\end{bmatrix}
\]

\[
\begin{bmatrix}
  3 & -3 & 1 & 0 \\
  -2 & 2 & 5 & 0 \\
  -1 & -1 & -2 & 0
\end{bmatrix}
\]

\[
\begin{bmatrix}
  1 & -4 & 7 & 2 \\
  0 & 3 & -5 & 1 \\
  0 & 5 & -9 & 0
\end{bmatrix}
\]