Assignment #3: Linear Independence, Linear Transformations, and Applications

Due date: Wednesday, February 17, 2016 (10:10am)
For full credit you must show all of your work.

1) Are the columns of the following matrices linearly independent? Explain how you know, in each case. You should be able to answer all three by inspection.
   a) \[
   \begin{bmatrix}
   87 & 93 & 31 \\
   -29 & 41 & 69
   \end{bmatrix}
   \]
   b) \[
   \begin{bmatrix}
   0 & 0 & 1 \\
   1 & 1 & 0 \\
   0 & 1 & 1
   \end{bmatrix}
   \]
   c) \[
   \begin{bmatrix}
   1 & 0 & 1 \\
   2 & 1 & 3 \\
   3 & 2 & 5 \\
   4 & 3 & 7
   \end{bmatrix}
   \]

2) For what values of \( h \) will the vectors in each of the following sets be linearly independent? Briefly explain your answer in each case.
   a) \[
   \begin{bmatrix}
   -1 \\
   1 \\
   1
   \end{bmatrix}, \begin{bmatrix}
   1 \\
   -1 \\
   h
   \end{bmatrix}, \begin{bmatrix}
   0
   \end{bmatrix}
   \]
   b) \[
   \begin{bmatrix}
   1 \\
   2 \\
   -1
   \end{bmatrix}, \begin{bmatrix}
   h \\
   4 \\
   -2
   \end{bmatrix}, \begin{bmatrix}
   1
   \end{bmatrix}
   \]
   c) \[
   \begin{bmatrix}
   3 \\
   1 \\
   1
   \end{bmatrix}, \begin{bmatrix}
   -1 \\
   3 \\
   3
   \end{bmatrix}
   \]

3) For each of the following statements, assume that the first clause is true and indicate whether the second clause will be always, sometimes or never true. Please justify your answers.
   a) \( \{v_1 \ldots v_m\} \) is a set of linearly independent vectors in \( \mathbb{R}^n \); \( n < m \).
   b) A homogeneous system \( Ax = 0 \) has the trivial solution; The columns of \( A \) are linearly dependent.
   c) Three vectors \( v_1, v_2 \) and \( v_3 \) are in \( \mathbb{R}^4 \) and \( v_3 \) is not a linear combination of \( v_1 \) and \( v_2 \); The set \( \{v_1, v_2, v_3\} \) is linearly independent.

4) For each of the following, give an example or explain why it is impossible to construct one:
   a) A \( 2 \times 2 \) coefficient matrix whose columns span \( \mathbb{R}^2 \)
   b) A \( 2 \times 3 \) coefficient matrix whose columns span \( \mathbb{R}^3 \)
   c) A \( 3 \times 2 \) coefficient matrix whose columns span \( \mathbb{R}^3 \)

5) Let \( A = \begin{bmatrix}
3 & 2 & 5 \\
-6 & 1 & -5 \\
2 & 4 & 6
\end{bmatrix} \). Observe that the third column is equal to the sum of the first and second.
   Without reducing the matrix, find a non-trivial solution to the homogeneous equation \( Ax = 0 \),
   \[
   \begin{bmatrix}
3 & 2 & 5 \\
-6 & 1 & -5 \\
2 & 4 & 6
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
x_3
\end{bmatrix} = \begin{bmatrix}
0 \\
0 \\
0
\end{bmatrix} = 0. \text{ Hint: consider } \begin{bmatrix}
3 \\
-6 \\
2
\end{bmatrix}x_1 + \begin{bmatrix}
2 \\
1 \\
4
\end{bmatrix}x_2 + \begin{bmatrix}
5 \\
-5 \\
6
\end{bmatrix}x_3 = 0
   \]

6) Give a geometric description of each of the following transformations \( x \rightarrow Ax \):
   a) \( A = \begin{bmatrix}
0 & -1 \\
1 & 0
\end{bmatrix} \)
   b) \( A = \begin{bmatrix}
0 & 1 \\
-1 & 0
\end{bmatrix} \)
   c) \( A = \begin{bmatrix}
1 & 0 \\
-1 & 1
\end{bmatrix} \)
7) Consider the linear transformation T achieved by the matrix \[
\begin{bmatrix}
a & b & c \\
d & e & f \\
\end{bmatrix}
\].

What is the *domain* of T? What is the *co-domain* of T?

8) If \( A = \begin{bmatrix} 2 & 2 & 0 \\ 0 & 5 & 7 \\ -3 & 2 & 0 \end{bmatrix} \) and \( x = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \), what is the *image* of x under the transformation \( T(x) = Ax \)?

9) If \( A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 0 & 1 & 0 \end{bmatrix} \), what is the *range* of the transformation \( T(x) = Ax \)?

10) Find all \( x \in \mathbb{R}^3 \) that are mapped onto the zero vector by the transformation \( x \rightarrow Ax \) where \( A = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -2 \end{bmatrix} \)

11) Suppose T is a linear transformation from \( \mathbb{R}^3 \) to \( \mathbb{R}^2 \) and \( T(e_1) = (1, -2), T(e_2) = (-2, 4) \) and \( T(e_3) = (0, 0) \) where \( e_1, e_2 \) and \( e_3 \) are the columns of the 3 \( \times \) 3 identity matrix. What is the *standard matrix* of the transformation T?

12) For each of the following linear transformations, specify whether it is one-to-one or onto:
   a) \[
   \begin{bmatrix}
   1 & -1 & 0 \\
   0 & 1 & -2 \\
   \end{bmatrix}
   \]
   b) \[
   \begin{bmatrix}
   1 & 0 \\
   2 & 0 \\
   1 & 1 \\
   \end{bmatrix}
   \]
   c) \[
   \begin{bmatrix}
   1 & 2 & 3 \\
   2 & 4 & 6 \\
   0 & 1 & 0 \\
   \end{bmatrix}
   \]

13) Acetylene (\( \text{C}_2\text{H}_2 \)) and oxygen (\( \text{O}_2 \)) can be combined in a chemical reaction to produce carbon dioxide (\( \text{CO}_2 \)) and water (\( \text{H}_2\text{O} \)).
   a) For each compound, construct a vector that lists the numbers of atoms of carbon (C), hydrogen (H), and oxygen (O)
   b) Show how the methods taught in this class can be used to determine the smallest integer weights that can be applied to each compound to balance the chemical equation \( \text{C}_2\text{H}_2 + \text{O}_2 \rightarrow \text{CO}_2 + \text{H}_2\text{O} \)

14) Suppose that one serving of Shredded Wheat supplies 160 calories, 4 grams of protein, 8 grams of fiber, and 2 grams of fat, and that one serving of Cheerios supplies 120 calories, 6g of protein, 0g of fiber, and 3g of fat.
   a) Suppose you want to create a new cereal that contains two parts Shredded Wheat and one part Cheerios. Set up a vector equation for this problem.
   b) How many servings of each cereal would you need to eat in order to consume exactly 5 grams of protein, 2 grams of fiber, and 2.5 grams of fat? How many calories would this be?
   c) Is it possible to combine Shredded Wheat and Cheerios to obtain a cereal that supplies 200 calories, 2g of protein, 16g of fiber, and 4g of fat? If so, in what proportions? If not, why not?