Assignment #10: Orthogonal Sets, Orthogonal Projections, Gram-Schmidt, and Least Squares

Due date: Wednesday, April 20, 2016 (10:10am)

Name: __________________________________________________________
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For full credit you must show all of your work.

1. For each of the following, choose the correct option:
   a. True or False: every matrix that has orthogonal columns is an orthogonal matrix
   b. True or False: every linearly independent set of vectors in $\mathbb{R}^n$ is an orthogonal set
   c. True or False: if two vectors $v_1$ and $v_2$ are orthogonal, then they are also linearly independent
   d. True or False: if two vectors $v_1$ and $v_2$ are orthogonal, and neither $v_1$ nor $v_2$ equals zero, then $v_1$ and $v_2$ are linearly independent
   e. True or False: if $A$ is an orthogonal matrix, then $A$ is invertible
   f. True or False: if $A$ is an orthogonal matrix, then $AA^T = I$
   g. True or False: if $A$ is a matrix that has orthogonal columns, then $A^T A = I$
   h. True or False: if $A$ and $B$ are each orthogonal matrices, then their product $AB$ is also an orthogonal matrix

2. What is the orthogonal projection of $x = \begin{bmatrix} -6 \\ 1 \\ 18 \end{bmatrix}$ onto $u = \begin{bmatrix} 0 \\ 3 \\ 4 \end{bmatrix}$?

3. Express $x = \begin{bmatrix} -6 \\ 1 \\ 18 \end{bmatrix}$ as the sum of a vector $\hat{x}$ that is contained within the subspace $W$ of $\mathbb{R}^3$ spanned by $u = \begin{bmatrix} 0 \\ 3 \\ 4 \end{bmatrix}$ and a vector $y$ that is contained within the subspace $W^\perp$. Verify that $\hat{x} \cdot y = 0$.

4. If $v_1 = \begin{bmatrix} 1 \\ 3 \\ 3 \\ 1 \end{bmatrix}$, $v_2 = \begin{bmatrix} 1 \\ -1 \\ -3 \end{bmatrix}$, $v_3 = \begin{bmatrix} 3 \\ 3 \\ -3 \\ 1 \end{bmatrix}$ and $v_4 = \begin{bmatrix} -3 \\ 1 \\ 1 \\ -3 \end{bmatrix}$ form an orthogonal basis for $\mathbb{R}^4$,
   a. Find the orthogonal projection of $x = \begin{bmatrix} -8 \\ 8 \\ 6 \\ 6 \end{bmatrix}$ onto each of the 1D subspaces of $\mathbb{R}^4$ spanned by each of the basis vectors $v_i$. 


b. Find the closest point \( \hat{x} \) to \( x \) in the subspace \( W \) of \( \mathbb{R}^4 \) spanned by \( \{v_1, v_2\} \)

c. Express \( x \) as the sum of two orthogonal vectors, \( u \) which is in the subspace \( W \) spanned by \( \{v_1, v_2\} \) and \( w \) which is in \( W^\perp \)

d. Express \( x = \begin{bmatrix} -8 \\ 8 \\ 6 \\ 6 \end{bmatrix} \) as a linear combination of \( \{v_1, v_2, v_3, v_4\} \).

5. Let \( u_1 = \begin{bmatrix} 2/3 \\ -2/3 \\ 1/3 \end{bmatrix} \) and \( u_2 = \begin{bmatrix} 1/3 \\ 2/3 \end{bmatrix} \), and let \( U = [u_1 \ u_2] \).

a. Compute \( A = U^T U \) and \( B = U U^T \).

b. Compute \( w = (U U^T) y \) where \( y = \begin{bmatrix} -1 \\ 7 \\ 7 \end{bmatrix} \).

c. Is it possible to express \( w \) as a linear combination of \( u_1 \) and \( u_2 \)? Is it possible to express \( y \) as a linear combination of \( u_1 \) and \( u_2 \)? How do you know?

d. Express \( y \) as the sum of two orthogonal vectors, one of which is in the subspace spanned by \( u_1 \) and \( u_2 \).

6. Use the Gram-Schmidt Process to find an orthogonal basis for the subspace of \( \mathbb{R}^4 \) spanned by

\[
\begin{align*}
\mathbf{u}_1 &= \begin{bmatrix} -1 \\ 3 \\ 1 \\ 1 \end{bmatrix}, \\
\mathbf{u}_2 &= \begin{bmatrix} 2 \\ 4 \\ 2 \\ 0 \end{bmatrix}, \\
\mathbf{u}_3 &= \begin{bmatrix} -10 \\ 4 \\ -4 \\ 6 \end{bmatrix}
\end{align*}
\]

7. What complications can arise when you try to use the Gram-Schmidt process to find an orthogonal basis for the column space of an \( m \times n \) matrix whose columns are linearly dependent? Try it on the following matrices:

\[
\begin{align*}
\mathbf{A} &= \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \end{bmatrix}, \\
\mathbf{B} &= \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}, \\
\mathbf{C} &= \begin{bmatrix} 1 & 2 & 2 \\ 0 & 0 & 1 \end{bmatrix}
\end{align*}
\]

What do you conclude about the applicability of the Gram-Schmidt process for finding an orthogonal basis for the column space of an arbitrary matrix?

8. Using Matlab, write a function \( \text{GramSchmidt()} \) that takes as input an arbitrary \( m \times n \) matrix \( \mathbf{M} \) and produces as output an \( m \times p \) matrix \( \mathbf{B} \) whose columns form an orthogonal basis for the subspace of \( \mathbb{R}^m \) spanned by the columns of \( \mathbf{M} \). Your program should not attempt to normalize the columns of \( \mathbf{B} \). Test your program on the input from questions 7 and 8 above and verify that your program produces the expected results. Please pass in a printout of your code and its output.
9. Working by hand, find the QR decomposition of \( A = \begin{bmatrix} 1 & 0.5 \\ 2 & 2 \\ -2 & 0 \end{bmatrix} \), where \( Q \) is a 3 x 2 matrix whose columns form an orthonormal basis for the column space of \( A \) and \( R \) is a 2 x 2 matrix such that \( A = QR \) (in other words, \( R = Q^T A \)).

10. Working by hand, find the QR decomposition of \( A = \begin{bmatrix} 1 & 0.5 \\ 2 & 2 \\ -2 & 0 \end{bmatrix} \), where \( Q \) is a 3 x 3 matrix whose columns form an orthonormal basis for \( \mathbb{R}^3 \) and \( R \) is a 3 x 2 matrix such that \( A = QR \) (in other words, \( R = Q^T A \)).

11. Use Householder reflections to derive the QR decomposition of \( A = \begin{bmatrix} 1 & 0.5 \\ 2 & 2 \\ -2 & 0 \end{bmatrix} \).

12. Find the least-squares fit of the following 4 points to a line: (0, 5), (1, 4) (–1, 0), (–2, –1). What is the least squares error in this approximation?

13. Which of the systems listed below has a unique least squares solution? Explain how you can tell without calculating the least squares solution.

\[
\begin{align*}
\begin{bmatrix}
1 & -1 & 1 \\
2 & -2 & 3 \\
2 & -2 & 3 \\
1 & -1 & 1
\end{bmatrix}
\begin{bmatrix}
\uparrow \\
x \\
\downarrow
\end{bmatrix}
&= 
\begin{bmatrix}
2 \\
1 \\
1 \\
2
\end{bmatrix} \\
\begin{bmatrix}
1 & 2 & 2 & 1 \\
1 & 3 & 3 & 1 \\
2 & 3 & 1 & 3
\end{bmatrix}
\begin{bmatrix}
\uparrow \\
x \\
\downarrow
\end{bmatrix}
&= 
\begin{bmatrix}
1 \\
2 \\
1 \\
0
\end{bmatrix} \\
\begin{bmatrix}
1 & 1 & 2 \\
0 & -2 & 1 \\
0 & 1 & 3 \\
0 & 0 & -1
\end{bmatrix}
\begin{bmatrix}
\uparrow \\
x \\
\downarrow
\end{bmatrix}
&= 
\begin{bmatrix}
0 \\
1 \\
2 \\
3
\end{bmatrix}
\end{align*}
\]