VECTORS [PARTS OF 1.3]
Vectors and the set $\mathbb{R}^n$

- A vector of dimension $n$ is an ordered list of $n$ numbers

**Example:**

$$v = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}; \quad w = \begin{bmatrix} 0 \\ 1 \end{bmatrix}; \quad z = \begin{bmatrix} 0 \\ 1 \\ -1 \\ 4 \end{bmatrix}.$$

- $v$ is in $\mathbb{R}^3$, $w$ is in $\mathbb{R}^2$ and $z$ is in $\mathbb{R}^4$.

- In $\mathbb{R}^3$ the $\mathbb{R}$ stands for the set of real numbers that appear as entries in the vector, and the exponents 3, indicate that each vector contains 3 entries.

- A vector can be viewed just as a matrix of dimension $m \times 1$. 
\( \mathbb{R}^n \) is the set of all vectors of dimension \( n \). We will see later that this is a vector space, i.e., a set that has some special properties with respect to operations on vectors.

Two vectors in \( \mathbb{R}^n \) are equal when their corresponding entries are all equal.

Given two vectors \( u \) and \( v \) in \( \mathbb{R}^n \), their sum is the vector \( u + v \) obtained by adding corresponding entries of \( u \) and \( v \).

Given a vector \( u \) and a real number \( \alpha \), the scalar multiple of \( u \) by \( \alpha \) is the vector \( \alpha u \) obtained by multiplying each entry in \( u \) by \( \alpha \).

(!) Note: the two vectors must be both in \( \mathbb{R}^n \), i.e., then both have \( n \) components.

Let us look at this in detail
**Sum of two vectors**

\[
x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}; \quad y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}; \quad \rightarrow \quad x + y = \begin{bmatrix} x_1 + y_1 \\ y_2 + x_2 \\ x_3 + y_3 \end{bmatrix}
\]

with numbers:

\[
x = \begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix}; \quad y = \begin{bmatrix} 0 \\ 3 \\ -3 \end{bmatrix}; \quad \rightarrow \quad x + y = \begin{bmatrix} -1 \\ 5 \\ ?? \end{bmatrix}
\]
**Multiplication by a scalar**

Given: a number \( \alpha \) (a 'scalar') and a vector \( x \):

\[
\alpha \in \mathbb{R}, \quad x \in \mathbb{R}^3, \quad \rightarrow \alpha x = \begin{bmatrix} \alpha x_1 \\ \alpha x_2 \\ \alpha x_3 \end{bmatrix}
\]

with numbers:

\[
\alpha = 4; \quad x = \begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix} \quad \rightarrow \alpha x = \begin{bmatrix} -4 \\ 8 \\ 12 \end{bmatrix}
\]

In the text vectors are represented by bold characters and scalars by light characters. We will often use Greek letters for scalars and regular latin symbols for vectors.
Properties of $+$ and $\alpha$*

- The vector whose entries are all zero is called the zero vector and is denoted by $0$.

  - (a) $x + y = y + x$ (Addition is commutative)
  - (b) $x + (y + z) = (x + y) + z$ (Addition is associative)
  - (c) $0 + x = x + 0 = x$, ($0$ is the vector of all zeros)
  - (d) $x + (-x) = -x + x = 0$ ($-x$ is the vector $(-1)x$)
  - (e) $\alpha(x + y) = \alpha x + \alpha y$
  - (f) $(\alpha + \beta)x = \alpha x + \beta x$
  - (g) $(\alpha\beta)x = \alpha(\beta x)$
  - (h) $1x = x$ for any $x$
**Linear combinations**

- Very important concept ..

A linear combination of $m$ vectors is a vector of the form:

$$x = \alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_m x_m$$

where $\alpha_1, \alpha_2, \cdots, \alpha_m$, are scalars and $x_1, x_2, \cdots, x_m$, are vectors in $\mathbb{R}^n$.

- The scalars $\alpha_1, \alpha_2, \cdots, \alpha_m$ are called the weights of the linear combination

- They can be any real numbers, including zero
**Linear combinations**

**Example:** Linear combinations of vectors in $\mathbb{R}^3$:

$$ u = 2 \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + 2 \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} ; \quad w = 2 \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} - \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 3 \end{bmatrix} $$

And we have:

$$ u = \begin{bmatrix} 2 \\ 0 \\ 4 \end{bmatrix} ; \quad w = \begin{bmatrix} ? \\ ? \\ ? \end{bmatrix} $$

**Note:** for $w$ the second weight is $-1$ and the third is $+1$. 
The linear span of a set of vectors

**Definition:** If \( v_1, \ldots, v_p \) are in \( \mathbb{R}^n \), then the set of all linear combinations of \( v_1, \ldots, v_p \) is denoted by \( \text{span}\{v_1, \ldots, v_p\} \) and is called the subset of \( \mathbb{R}^n \) spanned (or generated) by \( v_1, \ldots, v_p \). That is, \( \text{span}\{v_1, \ldots, v_p\} \) is the collection of all vectors that can be written in the form \( \alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_p v_p \) with \( \alpha_1, \alpha_2, \ldots, \alpha_p \) scalars.

- What is \( \text{span}\{u\} \) in \( \mathbb{R}^2 \) where \( u = \begin{bmatrix} 2 \\ 0 \end{bmatrix} \)?
- What is \( \text{span}\{v\} \) in \( \mathbb{R}^2 \) where \( v = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \)?
- What is \( \text{span}\{u, v\} \) in \( \mathbb{R}^2 \) with \( u, v \) given above?
Does the vector $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ belong to this $\text{span}\{u, v\}$?

Same question for the vector $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

What is $\text{span}\{u, v\}$ in $\mathbb{R}^3$ when:

$u = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$; $v = \begin{bmatrix} 0 \\ 2 \\ -1 \end{bmatrix}$?

Do the vectors:

$a = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$; $b = \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}$

belong to $\text{span}\{u, v\}$ found in the previous question?

Is $\text{span}\{u, v\}$ the same as $\text{span}\{v, u\}$?

Is $\text{span}\{u, v\}$ the same as $\text{span}\{2u, -3v\}$?
Geometric representation of $\mathbb{R}^2$ and $\mathbb{R}^3$

Consider a rectangular coordinate system in the plane. The illustration shows the vector

$$x = \begin{bmatrix} a \\ b \end{bmatrix}$$

with $a = 4$, $b = 2$.

Each point in the plane is determined by an ordered pair of numbers, so we identify a geometric point $(a, b)$ with the column vector $\begin{bmatrix} a \\ b \end{bmatrix}$.

We may regard $\mathbb{R}^2$ as the set of all points in the plane.
\[ \mathbb{R}^2 \]

\( \begin{align*}
&x_1 \text{ in the horizontal direction, } x_2 \text{ in vertical direction}
\end{align*} \)
Often we draw an oriented line from origin to the point: 

\[(2,1)\]

\[(-1,-1)\]
horizontal = $x_2$, vertical = $x_3$, back to front direction = $x_1$ (However some representations may differ). We will use this one.
Geometric interpretation of addition of 2 vectors

**First viewpoint:**

Think of moving ("rigidly") one of the vectors so its origin is at endpoint of the other vector. Then $x + y$ is the vector from origin to the end point of the shifted vector.
Second viewpoint:

\[ x + y \] corresponds to the fourth vertex of the parallelogram whose other three vertices are: O, \( x \), and \( y \)

Using the first viewpoint, show geometrically how to add the 3 vectors

\[
\begin{bmatrix}
1 \\
1
\end{bmatrix},
\begin{bmatrix}
2 \\
0
\end{bmatrix},
\text{ and }
\begin{bmatrix}
-1 \\
-2
\end{bmatrix}
\]
Let $v$ be a nonzero vector in $\mathbb{R}^3$. Then $\text{span}\{v\}$ is the set of all scalar multiples of $v$. This is also the set of points on the line in $\mathbb{R}^3$ through $v$ and $0$. (Figure 1.0 from text).
**Geometric interpretation of \( \text{span}\{u, v\} \)**

- Let \( u, v \) be two nonzero vectors in \( \mathbb{R}^3 \) with \( v \) not a multiple of \( u \).
- Then \( \text{span}\{u, v\} \) is the plane in \( \mathbb{R}^3 \) that contains \( u, v, \) and \( 0 \).
- In particular, \( \text{span}\{u, v\} \) contains the two lines \( \text{span}\{u\} \) and \( \text{span}\{v\} \).

(See also Figure 1.1 from text).
LINEAR INDEPENDENCE [1.7]
**Linear independence**

**Definition**
- The set \( \{v_1, \ldots, v_p\} \) is said to be linearly dependent if there exist weights \( c_1, \ldots, c_p \), not all zero, such that
  \[
  c_1v_1 + c_2v_2 + \cdots + c_pv_p = 0
  \]
- It is linearly independent otherwise.
- The above equation is called linear dependence relation among the vectors \( v_1, \cdots, v_p \).

- Another way to express dependence: A set of vectors is linearly dependent if and only if there is one vector among them which is a linear combination of all the others.

Prove this
Q: Why do we care about linear independence?

A: When expressing a vector \( \mathbf{x} \) as a linear combination of a system \( \{\mathbf{v}_1, \cdots, \mathbf{v}_p\} \) that is linearly dependent, then we can find a smaller system in which we can express \( \mathbf{x} \)

A dependent system is ‘redundant’

Let \( \mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \). Is \( \{\mathbf{v}_1\} \) linearly independent? [special case where \( p = 1 \)]

A system consisting of a nonzero vector [at least one nonzero entry] is always linearly independent: True - False?

Are the following systems linearly independent:

\[ \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}, \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -10 \\ 0 \end{bmatrix} \right\}, \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \end{bmatrix} \right\} \]
Let \( v_1 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} \); \( v_2 = \begin{bmatrix} 4 \\ 1 \\ 5 \end{bmatrix} \); \( v_3 = \begin{bmatrix} -2 \\ 3 \\ 1 \end{bmatrix} \);

(a) Determine if \( \{v_1, v_2, v_3\} \) is linearly independent

(b) If possible find a linear dependence relation among \( v_1, v_2, v_3 \).

**Solution:** We must determine if the system:

\[
x_1 \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} + x_2 \begin{bmatrix} 4 \\ 1 \\ 5 \end{bmatrix} + x_3 \begin{bmatrix} -2 \\ 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}
\]

has a nontrivial solution

**Note** Solution is trivial when \( x_1 = x_2 = x_3 = 0 \)
Augmented syst: | Echelon 1st step | Echelon 2nd step
---|---|---
\[
\begin{bmatrix}
1 & 4 & -2 & 0 \\
1 & 1 & 3 & 0 \\
2 & 5 & 1 & 0
\end{bmatrix}
\rightarrow
\begin{bmatrix}
1 & 4 & -2 & 0 \\
0 & -3 & 5 & 0 \\
0 & -3 & 5 & 0
\end{bmatrix}
\rightarrow
\begin{bmatrix}
1 & 4 & -2 & 0 \\
0 & -3 & 5 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}
\]

- This system is equivalent to original one.
- **Select** \(x_3 = 3\) (to avoid fractions) and back-solve for \(x_2\) (\(x_2 = 5\)), and \(x_1\), (\(x_1 = -14\))
- **Conclusion:** there is a nontrivial solution

- **NOT** independent

(b) Linear dependence relation: From above,

\[-14v_1 + 5v_2 + v_3 = 0\]
Note: Text uses the reduced echelon form instead of back-solving. [Result is clearly the same. Both solutions are OK]

- With the reduced row echelon form

\[
\begin{bmatrix}
1 & 0 & 14/3 & 0 \\
0 & 1 & -5/3 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}
\]

- \( x_1 = -(14/3)x_3; \quad x_2 = (5/3)x_3 \)

- select \( x_3 = 3 \) then \( x_2 = 5, x_1 = 14 \)

- Recall: \( x_1, x_2 \) are basic variables, and \( x_3 \) is free