Matrix operations: Matrix-vector product

- Product of the matrix $A$ by the vector $x$:

$$y = A \cdot x$$

$$\begin{bmatrix}
\beta_1 \\
\vdots \\
\beta_j \\
\vdots \\
\beta_n
\end{bmatrix} = \begin{bmatrix}
a_{11} & \cdots & a_{1j} & \cdots & a_{1n} \\
\vdots & \ddots & \vdots & \ddots & \vdots \\
a_{i1} & \cdots & a_{ij} & \cdots & a_{in} \\
\vdots & \ddots & \vdots & \ddots & \vdots \\
a_{m1} & \cdots & a_{mj} & \cdots & a_{mn}
\end{bmatrix} \begin{bmatrix}
\alpha_1 \\
\vdots \\
\alpha_j \\
\vdots \\
\alpha_n
\end{bmatrix}$$

$$= \alpha_1 a_1 + \alpha_2 a_2 + \cdots + \alpha_n a_n$$

- $x$, $y$ are vectors; $y$ is the result of $A \times x$.
- $a_1, a_2, \ldots, a_n$ are the columns of $A$

- Can get $i$-th component of the result $y$ without the others:

$$\beta_i = \alpha_1 a_{i1} + \alpha_2 a_{i2} + \cdots + \alpha_n a_{in}$$

**Example:** In the above example extract $\beta_2$

$$\beta_2 = (-2) \times 0 + (1) \times (-1) + (-3) \times (3) = -10$$

- Can compute $\beta_1, \beta_2, \ldots, \beta_m$ in this way.
- This is the 'row-wise' form of the 'matvec'

Matrix operations: Matrix-Matrix product

- When $A$ is $m \times n$, $B$ is $n \times p$, the product $AB$ of the matrices $A$ and $B$ is the $m \times b$ matrix defined as

$$AB = [Ab_1, Ab_2, \ldots, Ab_p]$$

- Each $Ab_j$ is a matrix-vector product: the product of $A$ by the $j$-th column of $B$. Matrix $AB$ has dimension $m \times p$

- Can use what we know on matvecs to perform the product

1. Column form – In words:

"The $j$-th column of $AB$ is a linear combination of the columns of $A$, with weights $b_{1j}, b_{2j}, \ldots, b_{nj}$" (entries of $j$-th col. of $B$)
Example: \( A = \begin{bmatrix} 1 & 2 & -1 \\ 0 & -1 & 3 \end{bmatrix} \) \( B = \begin{bmatrix} -2 & 1 \\ 1 & -2 \\ -3 & 2 \end{bmatrix} \) \( AB =? \)

\[ B = \begin{bmatrix} 1 & 2 & -1 \\ 0 & -1 & 3 \end{bmatrix} \begin{bmatrix} -2 \\ 1 \\ -3 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix} \]

First column has been computed before: it is equal to:
\((-2)\)*(col. 1 of \( A \)) + \((1)\) *(col. 2 of \( A \)) + \((-3)\) *(col. 3 of \( A \))

Second column is equal to:
\((1)\) *(col. 1 of \( A \)) + \((-2)\) *(col. 2 of \( A \)) + \((2)\) *(col. 3 of \( A \))

Fix \( j \) and run \( i \) \( \rightarrow \) column-wise form just seen

Fix \( i \) and run \( j \) \( \rightarrow \) row-wise form

Example: Get second row of \( AB \) in previous example.

\( c_{2j} = a_{21}b_{1j} + a_{22}b_{2j} + a_{23}b_{3j}, \quad j = 1, 2 \)

Can be read as: \[ c_{2} = a_{21}b_{1} + a_{22}b_{2} + a_{23}b_{3}, \] or in words:
row2 of \( C = a_{21} \) (row1 of \( B \)) + \( a_{22} \) (row2 of \( B \)) + \( a_{23} \) (row3 of \( B \))
\( = 0 \) (row1 of \( B \)) + \((-1)\) (row2 of \( B \)) + \((3)\) (row3 of \( B \))
\( = [-10 \quad 8] \)