LINEAR MAPPINGS [1.8]
A transformation or function or mapping from $\mathbb{R}^n$ to $\mathbb{R}^m$ is a rule which assigns to every $x$ in $\mathbb{R}^n$ a vector $T(x)$ in $\mathbb{R}^m$.

$\mathbb{R}^n$ is called the domain space of $T$ and $\mathbb{R}^m$ the image space or co-domain of $T$.

Notation:

$$T : \mathbb{R}^n \longrightarrow \mathbb{R}^m$$

$T(x)$ is the image of $x$ under $T$
Example: Take the mapping from $\mathbb{R}^2$ to $\mathbb{R}^3$:

$$T : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$$

$$x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \longrightarrow T(x) = \begin{pmatrix} x_1 + x_2 \\ x_1x_2 \\ x_1^2 + x_2^2 \end{pmatrix}$$

Example: Another mapping from $\mathbb{R}^2$ to $\mathbb{R}^3$:

$$T : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$$

$$x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \longrightarrow T(x) = \begin{pmatrix} x_1 + x_2 \\ x_1 - x_2 \\ x_1 + 5x_2 \end{pmatrix}$$

What is the main difference between these 2 examples?
A mapping $T$ is linear if:

(i) $T(u + v) = T(u) + T(v)$ for $u, v$ in the domain of $T$
(ii) $T(\alpha u) = \alpha T(u)$ for all $\alpha \in \mathbb{R}$, all $u$ in the domain of $T$

The mapping of the second example given above is linear - but not for the first one.

If a mapping is linear then $T(0) = 0$. (Why?)

A mapping is linear if and only if

$$T(\alpha u + \beta v) = \alpha T(u) + \beta T(v)$$

for all scalars $\alpha, \beta$ and all $u, v$ in the domain of $T$.

Prove this
Given an \( m \times n \) matrix \( A \), consider the special mapping:

\[
T : \mathbb{R}^n \longrightarrow \mathbb{R}^m \\
x \longrightarrow y = Ax
\]

Domain == ??; Image space == ??

From what we saw earlier ['Properties of the matrix-vector product'] such mappings are linear

As it turns out:

If \( T \) is linear, there exists a matrix \( A \) such that \( T(x) = Ax \) for all \( x \) in \( \mathbb{R}^n \)

In plain English: ‘A linear mapping can be represented by a matvec’

\( A \) is the representation of \( T \).
Let $A$ be a square matrix. Is the mapping $x \to x + Ax$ linear? If so find the matrix associated with it.

Same questions for the mapping $x \to Ax + \alpha x$ - where $\alpha$ is a scalar.

Express the following mapping from $\mathbb{R}^3$ to $\mathbb{R}^2$ in matrix/vector form:

\[
\begin{align*}
y_1 &= 2x_1 - x_2 + 1 \\
y_2 &= x_2 - x_3 - 2
\end{align*}
\]

Is this a linear mapping?

Read Section 1.9 and explore the notions of onto mappings (‘surjective’) and one-to-one mappings (‘injective’) in the text. You must at least know the definitions.

A mapping is onto if and only if ....

A mapping is one-to-one if and only if ....
MATRIX OPERATIONS [2.1]
If $A$ is an $m \times n$ matrix ($m$ rows and $n$ columns) – then the scalar entry in the $i$th row and $j$th column of $A$ is denoted by $a_{ij}$ and is called the $(i, j)$-entry of $A$. 

\[
\begin{pmatrix}
a_{11} & \cdots & a_{1j} & \cdots & a_{1n} \\
\vdots & & \vdots & & \vdots \\
a_{i1} & \cdots & \boxed{a_{ij}} & \cdots & a_{in} \\
\vdots & & \vdots & & \vdots \\
a_{m1} & \cdots & a_{mj} & \cdots & a_{mn}
\end{pmatrix} = A
\]
The number \( a_{ij} \) is the \( i \)th entry (from the top) of the \( j \)th column.

Each column of \( A \) is a list of \( m \) real numbers, which identifies a vector in \( \mathbb{R}^m \) called a column vector.

The columns are denoted by \( a_1, \ldots, a_n \), and the matrix \( A \) is written as \( A = [a_1, a_2, \ldots, a_n] \).
The diagonal entries in an $m \times n$ matrix $A$ are $a_{11}, a_{22}, a_{33}, \ldots$, and they form the main diagonal of $A$.

A diagonal matrix is a matrix whose nondiagonal entries are zero.

An important example is the $n \times n$ identity matrix, $I_n$ (each diagonal entry equals one) - Example:

$$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Another important matrix is the zero matrix (all entries are 0). It is denoted by $O$. 

7-10
Equality of two matrices: Two matrices $A$ and $B$ are equal if they have the same size (they are both $m \times n$) and if their entries are all the same.

$$a_{ij} = b_{ij} \text{ for all } i = 1, \cdots, m, \quad j = 1, \cdots, n$$

Sum of two matrices: If $A$ and $B$ are $m \times n$ matrices, then their sum $A + B$ is the $m \times n$ matrix whose entries are the sums of the corresponding entries in $A$ and $B$.

If we call $C$ this sum we can write:

$$c_{ij} = a_{ij} + b_{ij} \text{ for all } i = 1, \cdots, m, \quad j = 1, \cdots, n$$

$$\begin{bmatrix} 4 & 0 & 5 \\ 1 & 3 & 2 \end{bmatrix} + \begin{bmatrix} 3 & 1 & -3 \\ 0 & 2 & -2 \end{bmatrix} = ??; \quad \begin{bmatrix} 4 & 0 & 5 \\ 1 & 3 & 2 \end{bmatrix} + \begin{bmatrix} 1 & -3 \\ 2 & -2 \end{bmatrix} = ??$$
scalar multiple of a matrix

If \( r \) is a scalar and \( A \) is a matrix, then the scalar multiple \( rA \) is the matrix whose entries are \( r \) times the corresponding entries in \( A \).

\[
(\alpha A)_{ij} = \alpha a_{ij} \quad \text{for all } i = 1, \ldots, m, \quad j = 1, \ldots, n
\]

**Theorem** Let \( A, B, \) and \( C \) be matrices of the same size, and let \( \alpha \) and \( \beta \) be scalars. Then

- \( A + B = B + A \)
- \( (A + B) + C = A + (B + C) \)
- \( A + 0 = A \)
- \( \alpha(A + B) = \alpha A + \alpha B \)
- \( (\alpha + \beta)A = \alpha A + \beta A \)
- \( \alpha(\beta A) = (\alpha \beta)A \)

Prove all of the above equalities
**Matrix Multiplication**

- When a matrix $B$ multiplies a vector $x$, it transforms $x$ into the vector $Bx$.
- If this vector is then multiplied in turn by a matrix $A$, the resulting vector is $A(Bx)$.

Thus $A(Bx)$ is produced from $x$ by a composition of mappings—the linear transformations induced by $B$ and $A$. 
**Goal:** to represent this composite mapping as a multiplication by a single matrix, call it $C$ for now, so that

$$A(Bx) = Cx$$

Assume $A$ is $m \times n$, $B$ is $n \times p$, and $x$ is in $\mathbb{R}^p$.

Denote the columns of $B$ by $b_1, \cdots, b_p$ and the entries in $x$ by $x_1, \cdots, x_p$. Then:

$$Bx = x_1b_1 + \cdots + x_pb_p$$
By the linearity of multiplication by $A$:

\[ A(Bx) = A(x_1b_1) + \cdots + A(x_pb_p) = x_1Ab_1 + \cdots + x_pAb_p \]

The vector $A(Bx)$ is a linear combination of $Ab_1, \cdots, Ab_p$, using the entries in $x$ as weights.

In matrix notation, this linear combination is written as

\[ A(Bx) = [Ab_1, Ab_2, \cdots, Ab_p].x \]

Thus, multiplication by $[Ab_1, Ab_2, \cdots, Ab_p]$ transforms $x$ into $A(Bx)$.

Therefore the desired matrix $C$ is the matrix

\[ C = [Ab_1, Ab_2, \cdots, Ab_p] \]

Denoted by $AB$
**Definition:** If \( A \) is an \( m \times n \) matrix, and if \( B \) is an \( n \times p \) matrix with columns \( b_1, \cdots, b_p \), then the product \( AB \) is the matrix whose \( p \) columns are \( Ab_1, \cdots, Ab_p \). That is:

\[
AB = A[b_1, b_2, \cdots, b_p] = [Ab_1, Ab_2, \cdots, Ab_p]
\]

Important to remember that:

**Multiplication of matrices corresponds to composition of linear transformations.**

Operation count: How many operations are required to perform product \( AB \)?
Compute \( AB \) when
\[
A = \begin{bmatrix}
2 & -1 \\
1 & 3
\end{bmatrix} \quad B = \begin{bmatrix}
0 & 2 & -1 \\
1 & 3 & -2
\end{bmatrix}
\]

Compute \( AB \) when
\[
A = \begin{bmatrix}
2 & -1 & 2 & 0 \\
1 & -2 & 1 & 0 \\
3 & -2 & 0 & 0
\end{bmatrix} \quad B = \begin{bmatrix}
1 & -1 & -2 \\
0 & -2 & 2 \\
2 & 1 & -2 \\
-1 & 3 & 2
\end{bmatrix}
\]

Can you compute \( AB \) when
\[
A = \begin{bmatrix}
2 & -1 \\
1 & 3
\end{bmatrix} \quad B = \begin{bmatrix}
0 & 2 \\
1 & 3 \\
-1 & 4
\end{bmatrix}
\]
Row-wise matrix product

- Recall what we did with matrix-vector product to compute a single entry of the vector $Ax$.
- Can we do the same thing here? i.e., How can we compute the entry $c_{ij}$ of the product $AB$ without computing entire columns?
- Do this to compute entry $(2,2)$ in the first example above.
- Operation counts: Is more or less expensive to perform the matrix-vector product row-wise instead of column-wise?
Properties of matrix multiplication

**Theorem** Let \( A \) be an \( m \times n \) matrix, and let \( B \) and \( C \) have sizes for which the indicated sums and products are defined.

- \( A(BC) = (AB)C \) (associative law of multiplication)
- \( A(B + C) = AB + AC \) (left distributive law)
- \( (B + C)A = BA + CA \) (right distributive law)
- \( \alpha(AB) = (\alpha A)B = A(\alpha B) \) for any scalar \( \alpha \)
- \( I_mA = A = AI_n \) (product with identity)

If \( AB = AC \) then \( B = C \) (‘simplification’) : True-False?

If \( AB = 0 \) then either \( A = 0 \) or \( B = 0 \) : True or False?

\( AB = BA \) : True or false??
Square matrices. Matrix powers

- Important particular case when $n = m$ - so matrix is $n \times n$
- In this case if $x$ is in $\mathbb{R}^n$ then $y = Ax$ is also in $\mathbb{R}^n$
- $AA$ is also a square $n \times n$ matrix and will be denoted by $A^2$
- More generally, the matrix $A^k$ is the matrix which is the product of $k$ copies of $A$:
  \[
  A^1 = A; \quad A^2 = AA; \quad \cdots \quad A^k = \underbrace{A \cdots A}_{k \text{ times}}
  \]
- For consistency define $A^0$ to be the identity: $A^0 = I_n$
- $A^l \times A^k = A^{l+k}$ – Also true when $k$ or $l$ is zero.
### Transpose of a matrix

Given an \( m \times n \) matrix \( A \), the transpose of \( A \) is the \( n \times m \) matrix, denoted by \( A^T \), whose columns are formed from the corresponding rows of \( A \).

**Theorem** : Let \( A \) and \( B \) denote matrices whose sizes are appropriate for the following sums and products.

- \((A^T)^T = A\)
- \((A + B)^T = A^T + B^T\)
- \((\alpha A)^T = \alpha A^T\) for any scalar \( \alpha \)
- \((AB)^T = B^T A^T\)