LU factorization: Motivation

- Suppose we have to solve many linear systems
  \[ Ax = b^{(1)}, \ Ax = b^{(2)}, \ldots, \ Ax = b^{(p)} \]
  where matrix \( A \) is the same but the right-hand sides are different

- Can solve each of them by Gaussian Elimination separately → inefficient
  
  \[ \text{Cost?} \]

- Can get the inverse \( A^{-1} \) then each solution is of the form \( x^{(k)} = A^{-1}b^{(k)} \)
  
  \[ \text{Cost?} \ [\text{Using method based on rref seen in Lec. Notes 8}] \]

Best option: Exploit the “LU factorization of \( A \)”

Main result is this:

Gaussian elimination algorithm can provide as a by-product a *factorization* of \( A \) into the product of a lower triangular matrix \( L \) with ones on the diagonal, and an upper triangular matrix \( U \):

\[ A = LU \]

In addition:

This factorization is obtained at virtually no extra cost.

How would you solve systems with multiple right-hand sides using this? What does this approach cost?

Next: The LU factorization. Where does it come from and how to get it?

LU factorization – Revisiting GE

- We now ignore the right-hand side in GE

Recall: Gaussian elimination amounts to performing \( n - 1 \) successive Gaussian transformations, i.e., multiplications (to the left) by elementary matrices that have ones on the diagonal and the negatives of the multipliers in the column being eliminated.

- Set \( A_0 \equiv A \). Then – results of the \( n - 1 \) steps:

\[
\begin{align*}
A_1 &= E_1 A_0 \\
A_2 &= E_2 A_1 = E_2 E_1 A_0 \\
A_3 &= E_3 A_2 = E_3 E_2 E_1 A_0 \\
& \quad \vdots \\
A_{n-1} &= E_{n-1} E_{n-2} \cdots E_2 E_1 A_0
\end{align*}
\]
- $A_{n-1} \equiv U$ is an upper triangular matrix.
- We have $U = E_{n-1}E_{n-2} \cdots E_2E_1A$ or:
  \[ A = \underbrace{E_1^{-1}E_2^{-1}E_3^{-1} \cdots E_{n-1}^{-1}}_{L} U \equiv LU \]
- $E_1, E_2, \cdots, E_{n-1}$ are all lower triangular matrices with ones on the diagonal.
- Each $E_j^{-1}$ is lower triangular with ones on the diagonal. (Why?)
- $L = E_1^{-1}E_2^{-1}E_3^{-1} \cdots E_{n-1}^{-1}$ is lower triangular (Why?)
- $L$ has ones on the diagonal.

How do we get $L$?
- Could we use: $L = E_1^{-1}E_2^{-1}E_3^{-1} \cdots E_{n-1}^{-1}$? Too complex!
- There is a simpler way:

**Theorem.** Assume that Gaussian elimination can terminate (no division by zero) and let $U$ be the final triangular matrix obtained and $L$ the lower triangular matrix with $l_{ii} = 1$, and, for $i > k$, $l_{ik} = piv_{ik}$, the multiplier used to eliminate row $i$ in step $k$. Then: $A = LU$.
- $l_{kk} = 1$ and for $i \neq k$, $l_{ik} = \text{multiplier } a_{ik}/a_{kk}$ at $k$-th step of GE.
- The matrix $A$ is the product of a unit lower triangular matrix $L$ and an upper triangular matrix $U$.

In the end:
- $A = LU$ with:
  \[ L = E_1^{-1}E_2^{-1}E_3^{-1} \cdots E_{n-1}^{-1} \]
  \[ U = A_{n-1} \]
- Called the LU decomposition (or factorization) of $A$.

Notes:
- $L$ is Lower triangular, and has ones on the diagonal – We say that it is unit lower triangular
- $U$ is the last matrix into which $A$ is transformed from Gaussian elimination. It is upper triangular.
- We know how to get $U$ [last matrix in GE]
- The main issue now is: How can we get $L$?

**LU factorization - an example**

Example: Let $A = \begin{pmatrix} 4 & -2 & 2 \\ -2 & 5 & 3 \\ 2 & 3 & 9 \end{pmatrix}$
- **Step 1** of GE uses the multipliers $l_{21} = -1/2$, $l_{31} = 1/2$.
  - Resulting matrix: $A_1 = \begin{pmatrix} 4 & -2 & 2 \\ 0 & 4 & 4 \\ 0 & 4 & 8 \end{pmatrix}$
- **Step 2** of Gaussian Elimination uses the multiplier $l_{32} = 1$.
  - Resulting matrix $A_2 = \begin{pmatrix} 4 & -2 & 2 \\ 0 & 4 & 4 \\ 0 & 0 & 4 \end{pmatrix} \equiv U$
Therefore:

\[
L = \begin{pmatrix}
1 & 0 & 0 \\
-1/2 & 1 & 0 \\
1/2 & 1 & 1 \\
\end{pmatrix} \quad U = \begin{pmatrix}
4 & -2 & 2 \\
0 & 4 & 4 \\
0 & 0 & 4 \\
\end{pmatrix}
\]

Verify that \( A = LU \)

LU factorization of the matrix \( A = \begin{pmatrix}
2 & 4 & 6 \\
1 & 5 & 9 \\
1 & 0 & -12 \\
\end{pmatrix} \)

For the same \( A \) compute the 3rd column of \( A^{-1} \).