Multifrontal methods

- Start with the frontal method.

- Recall: Finite element matrix:

\[ A = \sum A^{[e]} \]

\( A^{[e]} \) = element matrix associated with element e.

- An old idea: Execute Gaussian elimination as the elements are being assembled

- This is called the frontal method

- Very popular among finite element users: saves storage
The origin: Frontal method

Elimination of $x_1$ creates an update matrix
Matrix has 3 parts:

1) Fully assembled (no longer modified)

2) Frontal matrix: undergoes assembly + updates

3) Remainder: not accessed yet.
Assembly tree:  - analogue to elimination tree

Can proceed from several incoupled elements at the same time  
→ multifrontal technique [Duff & Reid, 1983]
Assembly tree for Multifrontal Method

MULTIFRONTAL

X5

X4

X1

X2

X3

A B C

D E F

X5

1 2 3 4 5 6 7
Multifrontal methods: extension to general matrices

- Elimination tree replaces assembly tree
- Proceed in post-order traversal of elimination tree in order not to violate task dependencies.
- When a node is eliminated an update matrix is created.
- This matrix is passed to the parent which adds it to its frontal matrix.
- Requires a stack of pending update matrices
- Update matrices popped out as they are needed
- Often implemented with nested dissection-type ordering
- More complex than a left-looking algorithm
Update Matrix

Frontal Matrix

Update Matrix

U₁ =

3
7

U₂ =

3
9
\[ A_3 + U_1 + U_2 \]

Frontal Matrix

Update Matrix

\[ U_3 \]
Eliminating nodes 1 and 2:

What happens on matrix

\[
\begin{bmatrix}
1 & * & * & * \\
2 & * & & * \\
* & * & 3 & * & * & * \\
4 & * & * & * & * \\
5 & * & * & * & * \\
* & * & 6 & * & * \\
* & * & 7 & * & * \\
* & * & * & 8 & * \\
* & * & * & * & 9
\end{bmatrix}
\]

\( \leftarrow U_1(3, :) \leftarrow U_2(3, :) \)

\( \leftarrow U_1(7, :) \)

\( \leftarrow U_2(9, :) \)
Supernodes

Contiguous columns tend to inherit the pattern of the columns from they are updated → Many columns will have same sparsity pattern.

A supernode = a set of contiguous columns in the Cholesky factor $L$ which have the same sparsity pattern.

The set $\{j, j + 1, \ldots, j + s\}$ is a supernode if

$$NZ(L_{*,k}) = NZ(L_{*,k+1}) \cup \{k + 1\} \quad j \leq k < j + s$$

where $NZ(L_{*,k})$ is nonzero set of column $k$ of $L$. 

10-10 – Direct2
Supernodes
Other terms used: Mass elimination, indistinguishible nodes, active variables in front, subscript compression,...

➤ Idea is old but first suggested by S. Eisenstat for speeding up sparse codes on vector machines.

➤ Beneficial on most machines

➤ Gains come in part from savings in Gather-Scatter operations.