Basic relaxation schemes

- Relaxation methods: Jacobi, Gauss-Seidel, SOR
- Basic convergence results
- Optimal relaxation parameter for SOR
- See Chapter 4 of text for details.

Gauss-Seidel iteration for solving $Ax = b$:

- Corrects $j$-th component of current approximate solution, to zero the $j\text{-th}$ component of residual for $j = 1, 2, \cdots, n$.

Iteration matrices

Jacobi, Gauss-Seidel, SOR, & SSOR iterations are of the form

$$x^{(k+1)} = Mx^{(k)} + f$$

Jacobi:

$$M_{Jac} = D^{-1}(E + F) = I - D^{-1}A$$

Gauss-Seidel:

$$M_{GS} = (D - E)^{-1}F = I - (D - E)^{-1}A$$

Successive Overrelaxation (SOR):

$$M_{SOR} = (D - \omega E)^{-1}(\omega F + (1 - \omega)D) = I - (\omega^{-1}D - E)^{-1}A$$

Symmetric Successive Overrelaxation (SSOR):

$$M_{SSOR} = I - \omega(2 - \omega)(D - \omega F)^{-1}D(D - \omega E)^{-1}A$$
**General convergence result**

Consider the iteration: \( x^{(k+1)} = Gx^{(k)} + f \)

1. Assume that \( \rho(G) < 1 \). Then \( I - G \) is non-singular and \( G \) has a fixed point. Iteration converges to a fixed point for any \( f \) and \( x^{(0)} \).

2. If iteration converges for any \( f \) and \( x^{(0)} \) then \( \rho(G) < 1 \).

**Example: Richardson's iteration**

\[ x^{(k+1)} = x^{(k)} + \alpha(b - Ax^{(k)}) \]

Assume \( \Lambda(A) \subset \mathbb{R} \). When does the iteration converge?

**A few well-known results**

- Jacobi and Gauss-Seidel converge for diagonal dominant matrices, i.e., matrices such that
  \[ |a_{ii}| > \sum_{j \neq i} |a_{ij}|, i = 1, \ldots, n \]

- SOR converges for \( 0 < \omega < 2 \) for SPD matrices

- The optimal \( \omega \) is known in theory for an important class of matrices called 2-cyclic matrices or matrices with property A.

**An observation**

A matrix has property \( A \) if it can be (symmetrically) permuted into a \( 2 \times 2 \) block matrix whose diagonal blocks are diagonal.

\[ PAP^T = \begin{bmatrix} D_1 & E \\ E^T & D_2 \end{bmatrix} \]

- Let \( A \) be a matrix which has property \( A \). Then the eigenvalues \( \lambda \) of the SOR iteration matrix and the eigenvalues \( \mu \) of the Jacobi iteration matrix are related by
  \[ (\lambda + \omega - 1)^2 = \lambda \omega^2 \mu^2 \]

- The optimal \( \omega \) for matrices with property \( A \) is given by
  \[ \omega_{opt} = \frac{2}{1 + \sqrt{1 - \rho(B)^2}} \]

where \( B \) is the Jacobi iteration matrix.

- The iteration \( x^{(k+1)} = Mx^{(k)} + f \) is attempting to solve \( (I - M)x = f \). Since \( M \) is of the form \( M = I - P^{-1}A \) this system can be rewritten as
  \[ P^{-1}Ax = P^{-1}b \]

where for SSOR, we have

\[ P_{SSOR} = (D - \omega E)D^{-1}(D - \omega F) \]

referred to as the SSOR ‘preconditioning’ matrix.

In other words:

**Relaxation iter. \iff Preconditioned Fixed Point Iter.**