REORDERINGS FOR FILL-REDUCTION

GENERAL SPARSE MATRICES

- Minimal degree ordering
- Nested Dissection (ND) ordering
- Complexity of ND for model problems
Two broad types of orderings used:

- Minimal degree ordering + many variations
- Nested dissection ordering + many variations

Minimal degree ordering is easiest to describe:

At each step of GE, select next node to eliminate, as the node \( v \) of smallest degree. After eliminating node \( v \), update degrees and repeat.
Minimal Degree Ordering

At any step $i$ of Gaussian elimination define for any candidate pivot row $j$

\[
\text{Cost}(j) = (nz_c(j) - 1)(nz_r(j) - 1)
\]

where $nz_c(j) =$ number of nonzero elements in column $j$ of ‘active’ matrix, $nz_r(j) =$ number of nonzero elements in row $j$ of ‘active’ matrix.

- Heuristic: fill-in at step $j$ is $\leq \text{cost}(j)$
- Strategy: select pivot with minimal cost.
- Local, greedy algorithm
- Good results in practice.
Many improvements made over the years


<table>
<thead>
<tr>
<th>Min. Deg. Algorithm</th>
<th>Storage (words)</th>
<th>Order. time</th>
</tr>
</thead>
<tbody>
<tr>
<td>Final min. degree</td>
<td>1,181 K</td>
<td>43.90</td>
</tr>
<tr>
<td>Above w/o multiple elimn.</td>
<td>1,375 K</td>
<td>57.38</td>
</tr>
<tr>
<td>Above w/o elimn. absorption</td>
<td>1,375 K</td>
<td>56.00</td>
</tr>
<tr>
<td>Above w/o incompl. deg. update</td>
<td>1,375 K</td>
<td>83.26</td>
</tr>
<tr>
<td>Above w/o indistiguishible nodes</td>
<td>1,308 K</td>
<td>183.26</td>
</tr>
<tr>
<td>Above w/o mass-elimination</td>
<td>1,308 K</td>
<td>2289.44</td>
</tr>
</tbody>
</table>

▷ Results for a 180 × 180 9-point mesh problem
Since this article, many important developments took place.

In particular the idea of “Approximate Min. Degree” and “Approximate Min. Fill”, see


**First Idea:** Use quotient graphs

* Avoids elimination graphs which are not economical
* Elimination creates cliques
* Represent each clique by a node termed an *element* (recall FEM methods)
* No need to create fill-edges and elimination graph
* Still expensive: updating the degrees
Second idea: Multiple Minimum degree

* Many nodes will have the same degree. Idea: eliminate many of them simultaneously –

* Specifically eliminate independent set of nodes with same degree.

Third idea: Approximate Minimum degree

* Degree updates are expensive –

* Goal: To save time.

* Approach: only compute an approximation (upper bound) to degrees.

* Details are complicated and can be found in Tim Davis’ book

8-7 – order2
Nested Dissection Reordering (Alan George)

- Computer science ‘Divide-and-Conquer’ strategy.
- Best illustration: PDE finite difference grid.
- Easily described by using recursivity and by exploiting ‘separators’: ‘separate’ the graph in three parts, two of which have no coupling between them. The 3rd set (‘the separator’) has couplings with vertices from both of the first 2 sets.
- Key idea: dissect the graph; take the subgraphs and dissect them recursively.
- Nodes of separators always labeled last after those of the parents.
For regular $n \times n$ meshes, can show: fill-in is of order $n^2 \log n$ and computational cost of factorization is $O(n^3)$

How does this compare with a standard band solver?
Nested dissection for a small mesh
Nested dissection: cost for a regular mesh

- In 2-D consider an $n \times n$ problem, $N = n^2$
- In 3-D consider an $n \times n \times n$ problem, $N = n^3$

<table>
<thead>
<tr>
<th></th>
<th>2-D</th>
<th>3-D</th>
</tr>
</thead>
<tbody>
<tr>
<td>space (fill)</td>
<td>$O(N \log N)$</td>
<td>$O(N^{4/3})$</td>
</tr>
<tr>
<td>time (flops)</td>
<td>$O(N^{3/2})$</td>
<td>$O(N^2)$</td>
</tr>
</tbody>
</table>

- Significant difference in complexity between 2-D and 3-D
Nested dissection and separators

- Nested dissection methods depend on finding a good graph separator: \( V = T_1 \cup UT_2 \cup S \) such that the removal of \( S \) leaves \( T_1 \) and \( T_2 \) disconnected.

- Want: \( S \) small and \( T_1 \) and \( T_2 \) of about the same size.

- Simplest version of the graph partitioning problem.

**A theoretical result:**

If \( G \) is a planar graph with \( N \) vertices, then there is a separator \( S \) of size \( \leq \sqrt{N} \) such that \( |T_1| \leq 2N/3 \) and \( |T_2| \leq 2N/3 \).

In other words “Planar graphs have \( O(\sqrt{N}) \) separators”

- Many techniques for finding separators: Spectral, iterative swapping (K-L), multilevel (Metis), BFS, ...
The 2-D model problem

- 2-D finite difference mesh with $N$ vertices.

**Theorem:**
With natural ordering, resulting fill-in is $\Theta(N^{3/2})$.

**Theorem:**
With any ordering, resulting fill-in is $\Omega(N \log N)$.

**Theorem:**
With nested dissection ordering, resulting fill-in is $O(N \log N)$. 

Ordering techniques in practice

- In practice: Nested dissection (+ variants) is preferred for parallel processing

- Good implementations of Min. Degree algorithm work well in practice. Currently AMD and AMF are best known implementations/variants/

- Best practical reordering algorithms usually combine Nested dissection and min. degree algorithms.