Assignment #2: Vector and Matrix Equations, Solution Sets of Linear Systems, Linear Independence

Due date: Wednesday, January 31, 2018 (1:25pm)

Name: ________________________________

Section Number


1) Consider the following system of linear equations:

\[
\begin{align*}
3x_1 + 6x_2 - 5x_3 &= 8 \\
x_1 + x_2 + x_3 &= 5 \\
2x_1 + 4x_2 - 4x_3 &= 4
\end{align*}
\]

a) (5 points) Show, step-by-step, how to obtain the reduced echelon form of this matrix

b) (2 points) Describe the solution set of this system. Is it unique?

2) (6 points) For each of the augmented matrices below, indicate whether the represented linear system has zero, one, or an infinite number of solutions.

a) \[
\begin{bmatrix}
1 & 3 & 5 & 7 \\
2 & 4 & 4 & 8 \\
2 & 2 & 4 & 4
\end{bmatrix}
\]

b) \[
\begin{bmatrix}
3 & 2 & 1 \\
0 & 0 & 1 & 3
\end{bmatrix}
\]

c) \[
\begin{bmatrix}
1 & 3 & 5 & 7 \\
0 & 0 & 5 & 7 \\
1 & 2 & 4 & 4
\end{bmatrix}
\]

3) (9 points) For what values of \( h \) are these systems consistent? Please justify your answers.

a) \[
\begin{bmatrix}
2 & 3 & 1 \\
4 & 6 & h
\end{bmatrix}
\]

b) \[
\begin{bmatrix}
3 & 2 & 1 \\
1 & 2 & h
\end{bmatrix}
\]

c) \[
\begin{bmatrix}
1 & 2 & 1 \\
2 & h & 1
\end{bmatrix}
\]

4) (8 points) Suppose \( a, b, c, \) and \( d \) are non-zero constants. What relationship must hold among the numbers \( a, b, c, \) and \( d \) in order to guarantee that the system below is consistent for all possible values of \( f \) and \( g \)?

\[
\begin{align*}
ax_1 + bx_2 &= f \\
(cx_1 + dx_2 &= g)
\end{align*}
\]

5) (6 points) Use Matlab to convert the following augmented matrices into reduced echelon form, then describe the solution sets in each case, identifying the free variables if any:

\[
\begin{bmatrix}
3 & 0 & -1 & 1 \\
2 & -1 & 3 & 8 \\
-1 & 2 & 1 & 1
\end{bmatrix}
\]

\[
\begin{bmatrix}
-2 & 3 & -3 & -2 \\
2 & -5 & 5 & 2 \\
-3 & 5 & -5 & -3
\end{bmatrix}
\]

\[
\begin{bmatrix}
3 & 2 & -5 & 2 \\
1 & 1 & 5 & 1 \\
1 & 1 & -1 & 2
\end{bmatrix}
\]

6) (4 points) Do the following linear systems have the same solution set? Explain your answer.

\[
\begin{bmatrix}
3 & -2 & 1 & 4 \\
1 & 2 & -1 & 4 \\
5 & -3 & 2 & 7
\end{bmatrix}
\]

\[
\begin{bmatrix}
-1 & 6 & 0 & 4 \\
0 & 4 & -4 & 4 \\
-3 & 4 & -2 & -2
\end{bmatrix}
\]
7) (3 points) A system of linear equations that has fewer equations than unknowns is sometimes called an underdetermined system. Can such a system ever have a unique solution? Explain.

8) (3 points) A system of linear equations that has more equations than unknowns is sometimes called an overdetermined system. Can such a system ever be consistent? Explain.

9) (12 points) Suppose you are given the two points, (0, 1) and (1, 2).
   a) Using the formula: \( y = mx + b \), write two different equations in which \((x, y)\) represent the points that you have been given, and \(m\) (slope) and \(b\) (intercept) are the variables you seek to determine.
   b) The set of points that lie on the line that passes through both (0,1) and (1,2) can be represented as the solution set to a system of two equations in two unknowns, expressed in the form:
      \[
      a_{11}x_1 + a_{12}x_2 = b_1 \\
      a_{21}x_1 + a_{22}x_2 = b_2
      \]
      Re-write your equations from part a) in this form.
   c) Express this same system as a vector equation \(x_1v_1 + x_2v_2 = y\).
   d) Express this same system in the form \(Ax = b\), where \(A\) is a 2x2 matrix, and \(x\) and \(b\) are vectors.

10) (12 points) Consider the following statements with respect to a linear system \(Ax = b\):
    i) the system is consistent
    ii) the system is not consistent
    iii) the system may or may not be consistent, depending on the values in \(b\)
    iv) the solution is unique
    v) the solution, if it exists, is not unique
    vi) the solution, if it exists, may or may not be unique, depending on the values in \(A\)

    Indicate which two of the above six statements apply when the coefficient matrix \(A\):
    a) Has a pivot in every row
    b) Has at least one row that lacks a pivot
    c) Has a pivot in every column
    d) Has at least one column that lacks a pivot

    Be sure to consider \(m \times n\) matrices where \(m > n\) and \(m < n\) as well as \(m = n\).

    Please justify your answers to the questions a)–d) either with an explanation or with one or two examples in each case.

11) (3 points) Without using row reduction, fill in the blanks to make the following system consistent: (Hint: form the product \(Ax = b\))
    \[
    \begin{bmatrix}
    3 & -1 & 1 \\
    -1 & 2 & 1 \\
    1 & 2 & 3 \\
    \end{bmatrix}
    \begin{bmatrix}
    2 \\
    1 \\
    0 \\
    \end{bmatrix}
    =
    \begin{bmatrix}
    4 \\
    5 \\
    - \\
    \end{bmatrix}
    \]

12) (6 points) Give an example of two vectors in \(\mathbb{R}^3\) whose span is:
    a) a single point
    b) a line
    c) a plane
13) (6 points) Let:

\[
\begin{align*}
\mathbf{v}_1 &= \begin{bmatrix} 1 \\ -1 \\ 2 \\ 0 \end{bmatrix}, \\
\mathbf{v}_2 &= \begin{bmatrix} 2 \\ 1 \\ -1 \\ 0 \end{bmatrix}, \\
\mathbf{v}_3 &= \begin{bmatrix} 0 \\ -1 \\ 1 \\ 2 \end{bmatrix}
\end{align*}
\]

Without doing any computation, answer the following questions:

a) Do the three vectors \{ \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3 \} span \mathbb{R}^4? Justify your answer.

b) Do the three vectors \{ \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3 \} span \mathbb{R}^3? Justify your answer.

14) (6 points) Consider the linear system \( \mathbf{A} \mathbf{x} = \mathbf{b} \) where

\[
\mathbf{A} = \begin{bmatrix} 3 & 1 & 0 \\ -1 & 2 & -1 \\ 0 & 2 & -2 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix}.
\]

Can \( \mathbf{b} \) be represented as a linear combination of the columns of \( \mathbf{A} \)? If so, what are the weights?

15) (8 points) For each case below, construct an \( m \times n \) matrix \( \mathbf{A} \) with all non-zero entries such that the linear system \( \mathbf{A} \mathbf{x} = \mathbf{b} \) satisfies the stated conditions, or explain why such a matrix cannot be formed:

a) \( \mathbf{A} \) is \( 2 \times 3 \) and the linear system has a solution for every possible vector \( \mathbf{b} \)

b) \( \mathbf{A} \) is \( 2 \times 3 \) and there are some vectors \( \mathbf{b} \) for which the linear system does not have a solution

c) \( \mathbf{A} \) is \( 3 \times 2 \) and the linear system has a solution for every possible vector \( \mathbf{b} \)

d) \( \mathbf{A} \) is \( 3 \times 2 \) and there are some vectors \( \mathbf{b} \) for which the linear system does not have a solution

16) Let \( \mathbf{A} = \begin{bmatrix} 3 & 1 & 2 \\ 2 & 2 & 0 \\ 1 & 0 & 1 \end{bmatrix} \). Observe that the third column is equal to the difference of the first and second.

Without reducing the matrix, find a non-trivial solution \( \mathbf{x} \neq \mathbf{0} \) to the homogeneous equation

\[
\begin{bmatrix} 3 & 1 & 2 \\ 2 & 2 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}. \quad \text{Hint: consider} \quad \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} x_1 + \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} x_2 + \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} x_3 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \] (4 points).

17) (8 points) Express the solution set of the following system in parametric vector form:

\[
\begin{align*}
x_1 - x_2 + 2x_3 &= 2 \\
-2x_1 + 2x_2 &= -4 \\
2x_1 - 2x_2 + x_3 &= 4
\end{align*}
\]

18) (4 points) Characterize the solutions sets of these two linear systems. How do the solution sets differ?

\[
\begin{align*}
x_1 - 2x_2 + 3x_3 &= 0 \\
x_1 - 2x_2 + 3x_3 &= 1
\end{align*}
\]

19) (6 points) For each of the matrices below please indicate if the columns are linearly independent. Explain how you know, in each case. You should be able to answer all three by inspection.

\[
\begin{align*}
\mathbf{A} &= \begin{bmatrix} 1 & 1 & 2 \\ 2 & 0 & -1 \end{bmatrix}, \\
\mathbf{B} &= \begin{bmatrix} 0 & 0 & 2 \\ 2 & 0 & 2 \\ 0 & 2 & 0 \end{bmatrix}, \\
\mathbf{C} &= \begin{bmatrix} 1 & -1 & 2 \\ 2 & 1 & 1 \\ -1 & 0 & -1 \\ 0 & 2 & -2 \end{bmatrix}
\end{align*}
\]