Assignment #7: Eigenvalues, Eigenvectors, and Diagonalization

Due date: Friday, March 30, 2018 (1:35pm)

Name: _____________________________________________________________

Section Number
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For full credit you must show all of your work.

1. Is \( \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \) is an eigenvector of \( A = \begin{bmatrix} 5 & 2 & 3 \\ 3 & 8 & 3 \\ 2 & 0 & 4 \end{bmatrix} \)? Please show all of your work.

2. Is 3 an eigenvalue of \( A = \begin{bmatrix} 3 & -4 & 2 \\ 4 & 3 & 5 \\ 2 & 5 & 3 \end{bmatrix} \)? Please show all of your work. [Hint: check to see if 3 is a root of the characteristic equation of \( A \).]

3. Determine the eigenvalues of the following matrices. Hint: use the rules of thumb discussed in class to save effort where possible.

   a. \( \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix} \)
   
   b. \( \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \)
   
   c. \( \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 3 \\ 0 & 0 & 3 \end{bmatrix} \)

4. Determine the eigenvalues and eigenvectors of the following matrices. Hint: use the characteristic equation. Express your answer in terms of \( a, b, \) and \( c \).

   a. \( \begin{bmatrix} a & a \\ a & a \end{bmatrix} \)
   
   b. \( \begin{bmatrix} a & b \\ a & b \end{bmatrix} \)
   
   c. \( \begin{bmatrix} a & b \\ b & a \end{bmatrix} \)
   
   d. \( \begin{bmatrix} a & b \\ b & c \end{bmatrix} \)

5. If \( A = \begin{bmatrix} 3 & 0 & 0 \\ -3 & 4 & 9 \\ 0 & 0 & 3 \end{bmatrix} \), what are the eigenvalues and eigenvectors of \( A \)?

6. Compute the eigenvalues and eigenvectors of \( A = \begin{bmatrix} 0 & -1 & -1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix} \). Please show all of your work.

7. For each of the following statements, indicate whether it is True or False:

   a. If \( \lambda_1 \) and \( \lambda_2 \) are distinct eigenvalues of a 2x2 matrix \( A \), \( v_1 \) is any non-zero vector in the eigenspace of \( A \) corresponding to \( \lambda_1 \) and \( v_2 \) is any non-zero vector in the eigenspace of \( A \) corresponding to \( \lambda_2 \), then \( v_1 \) and \( v_2 \) will be linearly independent.

   b. If \( v_1 \) and \( v_2 \) are linearly independent eigenvectors of a matrix \( A \), then \( v_1 \) and \( v_2 \) are in different eigenspaces corresponding to distinct eigenvalues of \( A \).

   c. An \( n \times n \) matrix can have at most \( n \) real eigenvalues.
d. If $\lambda$ is a non-zero eigenvalue of an invertible matrix $A$ then $1/\lambda$ is an eigenvalue of $A^{-1}$.

e. Any square matrix $A$ has the same eigenvalues as its transpose $A^T$.

f. Any square matrix $A$ has the same eigenvectors as its transpose $A^T$.

g. If two matrices $A$ and $B$ are similar, then they will have the same eigenvalues.

h. If two matrices $A$ and $B$ have the same eigenvalues, then they are similar.

i. If two matrices $A$ and $B$ are similar, then they will have the same eigenvectors.

j. If two matrices $A$ and $B$ have the same determinant, then they will have the same eigenvalues.

k. If two matrices $A$ and $B$ have the same eigenvalues, then they will have the same determinant.

l. If two matrices $A$ and $B$ have the same eigenvalues, then they will have the same eigenvectors.

m. If $U$ is an echelon form of $A$ then the eigenvalues of $U$ will be the same as the eigenvalues of $A$.

8. For each of the following statements, indicate whether it is True or False:

a. If any vector in $\mathbb{R}^n$ can be expressed as a linear combination of the eigenvectors of $A$, then $A$ is diagonalizable.

b. If an $n \times n$ matrix $A$ is diagonalizable, then every vector in $\mathbb{R}^n$ will be an eigenvector of $A$.

c. If an $n \times n$ matrix $A$ is diagonalizable, then, taken together, the basis vectors of each of the eigenspaces of $A$ will span $\mathbb{R}^n$.

9. Let $A = \begin{bmatrix} 5 & 6 & 7 \\ 2 & 1 & 2 \\ -6 & -6 & -8 \end{bmatrix}$. Find matrices $D$ and $P$ such that $AP = PD$ where $P$ is invertible and $D$ is diagonal. For full credit you must show all of your work.

10. Give an example of a $2 \times 2$ matrix that is invertible but not diagonalizable, or explain why such a matrix does not exist. Give an example of a $2 \times 2$ matrix that is diagonalizable but not invertible, or explain why such a matrix does not exist.

11. Suppose the fixed population of a small state is divided into three groups: urban, suburban, and rural. If the total population is 1,000,000 and, every year, 10% of the city dwellers and 60% of the rural population move to the suburbs, while 80% of the suburban population flees to the countryside and 10% moves into the city, will the total number of inhabitants in each region eventually reach a steady state, and, if so, what will that number be?
12. Use the simple version of the PageRank algorithm presented in class, with a 10% dampening factor, to calculate the relative importance of each website in the following small network. What is the Markov matrix associated with this system? What is the importance vector you compute? List the websites in their order of importance. You should use Matlab to perform these calculations, but for full credit you need to show all of your work, including intermediate steps.

13. Use three iterations of the power method to estimate the largest eigenvalue and corresponding eigenvector of \( A = \begin{bmatrix} 2 & 4 \\ 1 & 5 \end{bmatrix} \), starting with \( x_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \) and ending with \( x_3 \). You may use Matlab to help with the arithmetic. Compare your estimates to the true values. For full credit, you must show all of your work and report each of the intermediate estimates \( x_1, x_2, \lambda_1, \lambda_2 \) as well as the final estimates \( x_3 \) and \( \lambda_3 \).

14. Use two iterations of the inverse power method to estimate the smallest eigenvalue and a corresponding eigenvector of \( A = \begin{bmatrix} 2 & 4 \\ 1 & 5 \end{bmatrix} \), starting with \( x_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \). You may use Matlab to help with the arithmetic. Compare your estimates to the true values. For full credit, you must show all of your work and report the intermediate estimates \( x_1, \lambda_1 \) as well as the final estimates \( x_2 \) and \( \lambda_2 \).

15. Use three iterations of the shifted inverse power method to estimate the middle eigenvalue and a corresponding eigenvector of \( A = \begin{bmatrix} -3 & 8 & -6 \\ -4 & 9 & -6 \\ 2 & -2 & 2 \end{bmatrix} \), starting with \( x_0 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \) and \( \alpha = 3 \).

You may use Matlab to help with the arithmetic. Compare your estimates to the true values.