DETERMINANTS CHAP. 3

Determinants: summary of main results

A determinant of an \( n \times n \) matrix is a real number associated with this matrix. Its definition is complex for the general case → We start with \( n = 2 \) and list important properties for this case.

- Determinant of a \( 2 \times 2 \) matrix:

\[
\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc
\]

- Notation: \( \det(A) \) or \( \det \begin{bmatrix} a & b \\ c & d \end{bmatrix} \)

Next we list the main properties of determinants. These properties are also true for \( n \times n \) case to be defined later.

\textcolor{blue}{Det(A) for \( n \times n \) case} can be defined from GE when permutation is not used: \( \det(A) = \) product of pivots in GE. More on this later.

Properties written for columns (easier to write) but are also true for rows

**Notation:** We let \( A = [u, v] \) columns \( u, \) and \( v \) are in \( \mathbb{R}^2 \).

1. If \( v = \alpha u \) then \( \det(A) = 0 \).

- Determinant of linearly dependent vectors is zero
- If any one column is zero then determinant is zero

2. Interchanging columns or rows:

\[
\det[v, u] = -\det[u, v]
\]

3. Linearity:

\[
\det[u, \alpha v + \beta w] = \alpha \det[u, v] + \beta \det[u, w]
\]

- \( \det(A) = \) linear function of each column (individually)
- \( \det(A) = \) linear function of each row (individually)
- What is the determinant \( \det[u, v + \alpha u]? \)

4. Determinant of transpose

\[
\det(A) = \det(A^T)
\]

5. Determinant of Identity

\[
\det(I) = 1
\]

6. Determinant of a diagonal:

\[
\det(D) = d_1 d_2 \cdots d_n
\]
7. Determinant of a triangular matrix (upper or lower)
   \[ \text{det}(T) = a_{11}a_{22} \cdots a_{nn} \]

8. Determinant of product of matrices [IMPORTANT]
   \[ \text{det}(AB) = \text{det}(A)\text{det}(B) \]

9. Consequence: Determinant of inverse
   \[ \text{det}(A^{-1}) = \frac{1}{\text{det}(A)} \]

   What is the determinant of \( \alpha A \) (for \( 2 \times 2 \) matrices)?
   
   What can you say about the determinant of a matrix which satisfies \( A^2 = I \)?

   Is it true that \( \text{det}(A + B) = \text{det}(A) + \text{det}(B) \)?

10. We will define \( 3 \times 3 \) determinants from \( 2 \times 2 \) determinants:
    \[
    \begin{vmatrix}
    a_{11} & a_{12} & a_{13} \\
    a_{21} & a_{22} & a_{23} \\
    a_{31} & a_{32} & a_{33}
    \end{vmatrix}
    = a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}
    \]

    This is an expansion of the det. with respect to its 1st row.

    1st term \( = a_{11} \times \text{det of matrix obtained by deleting 1st row and 1st column.} \)

    2nd term \( = -a_{12} \times \text{det of matrix obtained by deleting row 1 and column 2. Note the sign change.} \)

    3rd term \( = a_{13} \times \text{det of matrix obtained by deleting row 1 and column 3.} \)

11. Calculate \[
    \begin{vmatrix}
    2 & 3 & 0 \\
    1 & 2 & -1 \\
    -1 & 2 & 1
    \end{vmatrix}
    \]

    We will now generalize this definition to any dimension recursively. Need to define following notation.

    We will denote by \( A_{ij} \) is the \((n - 1) \times (n - 1)\) matrix obtained by deleting row \( i \) and column \( j \) from the matrix \( A \).

    **Example:** If \( A = \begin{bmatrix} 2 & 3 & 0 \\ 1 & 2 & -1 \\ -1 & 2 & 1 \end{bmatrix} \) Then: \( A_{11} = \begin{bmatrix} 2 & -1 \\ 2 & 1 \end{bmatrix} \); \( A_{12} = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \); \( A_{13} = \begin{bmatrix} 1 & 2 \\ -1 & 2 \end{bmatrix} \); \( A_{23} = \begin{bmatrix} 2 & 3 \\ -1 & 2 \end{bmatrix} \)

12. Definition The determinant of a matrix \( A = [a_{ij}] \) is the sum
    \[ \text{det}(A) = + a_{11} \text{det}(A_{11}) - a_{12} \text{det}(A_{12}) + a_{13} \text{det}(A_{13}) - a_{14} \text{det}(A_{14}) + \cdots + (-1)^{1+n} a_{1n} \text{det}(A_{1n}) \]

    Note the alternating signs

    We can write this as:
    \[ \text{det}(A) = \sum_{j=1}^{n} (-1)^{1+j} a_{1j} \text{det}(A_{1j}) \]

    This is an expansion with respect to the 1st row.
Generalization: Cofactors

Define \( c_{ij} = (-1)^{i+j} \text{det } A_{ij} \) = cofactor of entry \( i,j \)

> Then we get a more general expansion formula:

\[ \text{det}(A) \text{ can be expanded with respect to } i\text{-th as follows } \]

\[ \text{det}(A) = a_{i1}c_{i1} + a_{i2}c_{i2} + \cdots + a_{in}c_{in} \]

> Note \( i \) is fixed. Can be done for any \( i \) [same result each time]

> Case \( i = 1 \) corresponds to definition given earlier

Similar expressions for expanding w.r.t. column \( j \) (now \( j \) is fixed)

\[ \text{det}(A) = a_{1j}c_{1j} + a_{2j}c_{2j} + \cdots + a_{nj}c_{nj} \]

Inverse of a matrix

Let \( B \) be the matrix obtained from a matrix \( A \) by multiplying a certain row (or column) of \( A \) by a scalar \( \alpha \). Use the definition to show that: \( \text{det}(B) = \alpha \text{det}(A) \).

What is the computational cost of evaluating the determinant using co-cofactor expansions? [Hint: It is BIG!]

Compute \( F_2, F_3, F_4 \) when \( F_n \) is the \( n \)-th dimensional determinant:

\[ F_n = \begin{bmatrix} 1 & -1 & 1 & -1 & \cdots & 1 & -1 \\ 1 & 1 & -1 & \cdots & 1 & 1 & -1 \\ \end{bmatrix} \]

(continuation) Challenge: Show a recurrence relation between \( F_n, F_{n-1} \) and \( F_{n-2} \). Do you recognize this relation? Compute the first 8 values of \( F_n \).

Computing determinants using cofactors

Compute the following determinant by using co-factors. Expand with respect to 1st row.

\[ \begin{vmatrix} -1 & 2 & 0 \\ 2 & -1 & 3 \\ -1 & 0 & 2 \end{vmatrix} \]

Compute the above determinant by using co-factors. Expand with respect to last row. Then expand with respect to last column.

Compute the following determinant [expand with respect to last row!]

\[ \begin{vmatrix} -1 & 2 & 0 & 1 \\ 2 & -1 & 3 & 2 \\ -1 & 0 & 2 & 3 \\ 0 & -1 & 0 & 3 \end{vmatrix} \]

Suppose two rows of \( A \) are swapped. Use the above definition to show that the determinant changes signs.

Let \( B \) be the matrix obtained from a matrix \( A \) by multiplying a certain row (or column) of \( A \) by a scalar \( \alpha \). Use the definition to show that:

\( \text{det}(B) = \alpha \text{det}(A) \).

Find the inverses of:

\[ \begin{bmatrix} 2 & 2 \\ -1 & 1 \end{bmatrix} \text{ and } \begin{bmatrix} 2 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -2 \end{bmatrix} \]

Inverse of a matrix

Let \( C = \{c_{ij}\} \) the matrix of cofactors. Entry \((i,j)\) of \( C \) has cofactor \( c_{ij} \). Then it is easy to prove that:

\[ A^{-1} = \frac{1}{\text{det}(A)} C^T \]

Example: Inverse of a \( 2 \times 2 \) matrix:

\[ \begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \]
**Areas in** \( \mathbb{R}^2 \)

**Left figure:** Area of a parallelogram spanned by points \((0, 0), (a, b), (c, d), (a+c, b+d)\) is:

\[
\text{det} \begin{bmatrix} a & c \\ b & d \end{bmatrix}
\]

**Right figure:** Area of triangle spanned by the points \((x_1, y_1), (x_2, y_2), (x_3, y_3)\) is:

\[
\frac{1}{2} \left| \begin{array}{ccc} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{array} \right|
\]

**Volumes in** \( \mathbb{R}^3 \)

**Volume** of parallelepiped spanned by origin and the 3 points \((x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3)\) is:

\[
\text{det} \begin{bmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{bmatrix}
\]

**In summary:** Volume \( (\mathbb{R}^3) \)/area \( (\mathbb{R}^2) \) of a box is \(|\text{det}(A)|\) when the box edges are the rows of \(A\).

**Areas, Volumes, and Mappings**

- Determinants are all about areas/volumes – Text has a lot more detail
- See section “Determinants as area or volume” in text
- See example 4 in same section
- Linear mappings and determinants [p. 184 in text]

**Important point:** Results also true for any region in \( \mathbb{R}^2 \) (1st part) and \( \mathbb{R}^3 \) (2nd part)

See Example 4 in Section 3.2 which uses this to compute the area of an ellipse.

**Theorem** Let \( T \) the linear mapping from/to \( \mathbb{R}^2 \) represented by a matrix \(A\). If \( S \) is a parallelogram in \( \mathbb{R}^2 \) then

\[
\{\text{area of } T(S) \} = |\text{det}(A)| \cdot \{\text{area of } S \}
\]

Similarly, if \( T \) is the linear mapping from/to \( \mathbb{R}^3 \) represented by a matrix \(A\) and \( S \) is a parallelipiped in \( \mathbb{R}^3 \) then

\[
\{\text{volume of } T(S) \} = |\text{det}(A)| \cdot \{\text{volume of } S \}
\]

- Important point: Results also true for any region in \( \mathbb{R}^2 \) (1st part) and \( \mathbb{R}^3 \) (2nd part)
- See Example 4 in Section 3.2 which uses this to compute the area of an ellipse.
How to compute determinants in practice?

- Co-factor expansion?? *Not practical*. Instead:
- Perform an LU factorization of $A$ with pivoting.
- If a zero column is encountered LU fails but $\det(A) = 0$
- If not get $\det = \text{product of diagonal entries multiplied by a sign } \pm 1$ depending on how many times we interchanged rows.

Compute the determinants of the matrices

$A = \begin{bmatrix} 2 & 4 & 6 \\ 1 & 5 & 9 \\ 1 & 0 & -12 \end{bmatrix}$  
$B = \begin{bmatrix} 0 & -1 & 1 & 2 \\ 1 & -2 & -1 & 1 \\ 2 & 0 & 2 & 0 \\ -1 & 1 & -1 & -1 \end{bmatrix}$