APPLICATION: ROTATION AND TRANSLATIONS [2.7]
In the form of exercises. Try to answer all questions before class [see textbook for help]

Consider the mapping that sends any point \( x \) in \( \mathbb{R}^2 \) into a point \( y \) in \( \mathbb{R}^2 \) that is rotated from \( x \) by an angle \( \theta \). Is the mapping linear?

Find the matrix representing the mapping. [Hint: observe how the canonical basis is transformed.]

[See Example 4 in Sect. 5.7 of text, See HW-2,..]
Solution: [see a previous HW]

See how $e_1$ are $e_2$ are changed.

- $e_1$ becomes $a_1 = \begin{bmatrix} \cos \theta \\ \sin \theta \end{bmatrix}$
- $e_2$ becomes $a_2 = \begin{bmatrix} \cos(\pi/2 + \theta) \\ \sin(\pi/2 + \theta) \end{bmatrix} = \begin{bmatrix} -\sin \theta \\ \cos \theta \end{bmatrix}$

The columns of $A$ are $a_1, a_2$; Therefore:

$$A = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}$$
Rotations and translations in $\mathbb{R}^2$

- Another very important operation: Translation or shift
- Recall: Not a linear mapping – but called affine mapping
- This will require a little artifice.

How can you now represent a translation via a matrix-vector product? [Hint: add an artificial component of 1 at the end of vector $x$]

- Called Homogeneous coordinates
- See Example 4 of Sect. 2.7 of *text* and then Example 6.
Solution: Call $f = [f_1; f_2]$ the translation vector

Let $\hat{x} = \begin{bmatrix} x_1 \\ x_2 \\ 1 \end{bmatrix}$;
Also write resulting vector $\hat{y}$ similarly as: $\hat{y} = \begin{bmatrix} y_1 \\ y_2 \\ 1 \end{bmatrix}$

We want: $\hat{y}_1 = x_1 + f_1$, $\hat{y}_2 = x_2 + f_2$

Then the matrix is clearly:

$$A = \begin{bmatrix} 1 & 0 & f_1 \\ 0 & 1 & f_2 \\ 0 & 0 & 1 \end{bmatrix}$$

Indeed, we do have:

$$\begin{bmatrix} 1 & 0 & f_1 \\ 0 & 1 & f_2 \\ 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} x_1 \\ x_2 \\ 1 \end{bmatrix} = \begin{bmatrix} x_1 + f_1 \\ x_2 + f_2 \\ 1 \end{bmatrix} = \hat{y}$$
Rotations and translations in $\mathbb{R}^2$

The most important mapping in real life is a combination of Rotation and Translation.

Find a mapping that combines rotation followed by translation

Hint: use the Homogeneous coordinates introduced above

Solution:

1. Rotation: Since this must leave the 1 at end of $\hat{x}$ unchanged, the matrix is

$$R = \begin{bmatrix}
\cos \theta & -\sin \theta & 0 \\
\sin \theta & \cos \theta & 0 \\
0 & 0 & 1
\end{bmatrix}$$
2. **Translation:** The translation matrix is (see above)

\[
T = \begin{bmatrix}
1 & 0 & f_1 \\
0 & 1 & f_2 \\
0 & 0 & 1
\end{bmatrix}
\]

3. **Compound the two:** This corresponds to product of matrices!

\[
A = TR = \begin{bmatrix}
\cos \theta & -\sin \theta & f_1 \\
\sin \theta & \cos \theta & f_2 \\
0 & 0 & 1
\end{bmatrix}
\]
Does the order matter? Reason from the geometry and then from the derivation of your matrix

One more operation: scaling by a weight $\alpha$ for example $\alpha = 0.3$. This corresponds to simply multiplying all coordinates by $\alpha$.

See Composite transformations in text. See Example 6 in Sec. 2.7 in text. Implement the example in matlab [represent the triangle with vertices $a= (-1, -1)$, $b = (1, -1)$, $c = (0,1)$. Ignore shading]
Practice. Continuing with Example 6 from [previous exercise.] Generate the following figure using what you just learned.

Details: Scaling = 0.9; Rotation angle: $\theta = \pi/12$; Translation vector $(0.9, -0.9)$. Repeat: 30 times.

Challenge question: The triangles seem to vanish into a limit point. What is this point?