VECTORS [PARTS OF 1.3]
Vectors and the set $\mathbb{R}^n$

- A vector of dimension $n$ is an ordered list of $n$ numbers

**Example:**

\[ v = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}; \quad w = \begin{bmatrix} 0 \\ 1 \end{bmatrix}; \quad z = \begin{bmatrix} 0 \\ 1 \\ -1 \\ 4 \end{bmatrix}. \]

- $v$ is in $\mathbb{R}^3$, $w$ is in $\mathbb{R}^2$ and $z$ is in $\mathbb{R}^4$?

- In $\mathbb{R}^3$ the $\mathbb{R}$ stands for the set of real numbers that appear as entries in the vector, and the exponents 3, indicate that each vector contains 3 entries.

- A vector can be viewed just as a matrix of dimension $m \times 1$
\( \mathbb{R}^n \) is the set of all vectors of dimension \( n \). We will see later that this is a vector space, i.e., a set that has some special properties with respect to operations on vectors.

Two vectors in \( \mathbb{R}^n \) are equal when their corresponding entries are all equal.

Given two vectors \( u \) and \( v \) in \( \mathbb{R}^n \), their sum is the vector \( u + v \) obtained by adding corresponding entries of \( u \) and \( v \).

Given a vector \( u \) and a real number \( \alpha \), the scalar multiple of \( u \) by \( \alpha \) is the vector \( \alpha u \) obtained by multiplying each entry in \( u \) by \( \alpha \).

(!) Note: the two vectors must be both in \( \mathbb{R}^n \), i.e., then both have \( n \) components.

Let us look at this in detail.
Sum of two vectors

\[ x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \; ; \; y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} \; ; \; \rightarrow \; x + y = \begin{bmatrix} x_1 + y_1 \\ y_2 + x_2 \\ x_3 + y_3 \end{bmatrix} \]

with numbers:

\[ x = \begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix} \; ; \; y = \begin{bmatrix} 0 \\ 3 \\ -3 \end{bmatrix} \; ; \; \rightarrow \; x + y = \begin{bmatrix} -1 \\ 5 \\ ?? \end{bmatrix} \]
Multiplication by a scalar

Given: a number $\alpha$ (a 'scalar') and a vector $x$:

$$\alpha \in \mathbb{R}, \quad x \in \mathbb{R}^3, \quad \rightarrow \alpha x = \begin{bmatrix} \alpha x_1 \\ \alpha x_2 \\ \alpha x_3 \end{bmatrix}$$

with numbers:

$$\alpha = 4; \quad x = \begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix} \quad \rightarrow \alpha x = \begin{bmatrix} -4 \\ 8 \\ 12 \end{bmatrix}$$

In the text vectors are represented by bold characters and scalars by light characters. We will often use Greek letters for scalars and regular latin symbols for vectors.
Properties of $+$ and $\alpha$*

The vector whose entries are all zero is called the zero vector and is denoted by 0.

- (a) $x + y = y + x$ (Addition is commutative)
- (b) $x + (y + z) = (x + y) + z$ (Addition is associative)
- (c) $0 + x = x + 0 = x$, ($0$ is the vector of all zeros)
- (d) $x + (-x) = -x + x = 0$ ($-x$ is the vector $(-1)x$)
- (e) $\alpha(x + y) = \alpha x + \alpha y$
- (f) $(\alpha + \beta)x = \alpha x + \beta x$
- (g) $(\alpha \beta)x = \alpha(\beta x)$
- (h) $1x = x$ for any $x$
Very important concept..

A linear combination of \( m \) vectors is a vector of the form:

\[ x = \alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_m x_m \]

where \( \alpha_1, \alpha_2, \cdots, \alpha_m \) are scalars and \( x_1, x_2, \cdots, x_m \) are vectors in \( \mathbb{R}^n \).

The scalars \( \alpha_1, \alpha_2, \cdots, \alpha_m \) are called the weights of the linear combination.

They can be any real numbers, including zero.
Linear combinations

**Example:** Linear combinations of vectors in $\mathbb{R}^3$:

$$u = 2 \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + 2 \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}; \quad w = 2 \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} - \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

And we have:

$$u = \begin{bmatrix} 2 \\ 0 \\ 4 \end{bmatrix}; \quad w = \begin{bmatrix} ? \\ ? \\ ? \end{bmatrix}$$

**Note:** for $w$ the second weight is $-1$ and the third is $+1$. 
**The linear span of a set of vectors**

**Definition:** If $v_1, \ldots, v_p$ are in $\mathbb{R}^n$, then the set of all linear combinations of $v_1, \ldots, v_p$ is denoted by $\text{span}\{v_1, \ldots, v_p\}$ and is called the subset of $\mathbb{R}^n$ spanned (or generated) by $v_1, \ldots, v_p$. That is, $\text{span}\{v_1, \ldots, v_p\}$ is the collection of all vectors that can be written in the form $\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_p v_p$ with $\alpha_1, \alpha_2, \cdots, \alpha_p$ scalars.

- What is $\text{span}\{u\}$ in $\mathbb{R}^2$ where $u = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$?
- What is $\text{span}\{v\}$ in $\mathbb{R}^2$ where $v = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$?
- What is $\text{span}\{u, v\}$ in $\mathbb{R}^2$ with $u, v$ given above?
Does the vector \(\begin{bmatrix} -1 \\ 1 \end{bmatrix}\) belong to this \(\text{span}\{u, v\}\)?

Same question for the vector \(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\).

What is \(\text{span}\{u, v\}\) in \(\mathbb{R}^3\) when:

\[
\begin{align*}
u &= \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} ; \\
v &= \begin{bmatrix} 0 \\ 2 \\ -1 \end{bmatrix}
\end{align*}
\]

Do the vectors:

\[
\begin{align*}
a &= \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} ; \\
b &= \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}
\end{align*}
\]

belong to \(\text{span}\{u, v\}\) found in the previous question?

Is \(\text{span}\{u, v\}\) the same as \(\text{span}\{v, u\}\) ?

Is \(\text{span}\{u, v\}\) the same as \(\text{span}\{2u, -3v\}\) ?
Consider a rectangular coordinate system in the plane. The illustration shows the vector

\[ x = \begin{bmatrix} a \\ b \end{bmatrix} \]

with \( a = 4, b = 2 \).

Each point in the plane is determined by an ordered pair of numbers, so we identify a geometric point \((a, b)\) with the column vector \( \begin{bmatrix} a \\ b \end{bmatrix} \).

We may regard \( \mathbb{R}^2 \) as the set of all points in the plane.
\( \mathbb{R}^2 \)

\[ (2,1) \]

\[ (-1,-1) \]

\[ x_1 \text{ in the horizontal direction}, \ x_2 \text{ in vertical direction} \]
Often we draw an oriented line from origin to the point:

(2,1)

(−1,−1)
horizontal = $x_2$, vertical = $x_3$, back to front direction = $x_1$ (However some representations may differ). We will use this one.
**Geometric interpretation of addition of 2 vectors**

**First viewpoint:**

Think of moving ("rigidly") one of the vectors so its origin is at endpoint of the other vector. Then $x + y$ is the vector from origin to the end point of the shifted vector.
**Second viewpoint:**

\( \mathbf{x} + \mathbf{y} \) corresponds to the fourth vertex of the parallelogram whose other three vertices are: \( \mathbf{O}, \mathbf{x}, \) and \( \mathbf{y} \)

Using the first viewpoint, show geometrically how to add the 3 vectors

\[
\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \end{bmatrix}, \text{ and } \begin{bmatrix} -1 \\ -2 \end{bmatrix}
\]
Let $\mathbf{v}$ be a nonzero vector in $\mathbb{R}^3$.

Then $\text{span}\{\mathbf{v}\}$ is the set of all scalar multiples of $\mathbf{v}$.

This is also the set of points on the line in $\mathbb{R}^3$ through $\mathbf{v}$ and $\mathbf{0}$. 

**Geometric interpretation of $\text{span}\{\mathbf{v}\}$**
Let $u, v$ be two nonzero vectors in $\mathbb{R}^3$ with $v$ not a multiple of $u$.

Then $\text{span}\{u, v\}$ is the plane in $\mathbb{R}^3$ that contains $u, v,$ and $0$.

In particular, $\text{span}\{u, v\}$ contains the two lines $\text{span}\{u\}$ and $\text{span}\{v\}$.

(See also Figure 1.1 from text).
LINEAR INDEPENDENCE [1.7]
Linear independence [Important]

Definition

- The set \( \{v_1, ..., v_p\} \) is said to be linearly dependent if there exist weights \( c_1, ..., c_p \), not all zero, such that

\[
c_1v_1 + c_2v_2 + ... + c_pv_p = 0
\]

- It is linearly independent otherwise

- The above equation is called linear dependence relation among the vectors \( v_1, ..., v_p \)

- Another way to express dependence: A set of vectors is linearly dependent if and only if there is one vector among them which is a linear combination of all the others.

Prove this
Q: Why do we care about linear independence?

A: When expressing a vector \( \mathbf{x} \) as a linear combination of a system \( \{ \mathbf{v}_1, \ldots, \mathbf{v}_p \} \) that is linearly dependent, then we can find a smaller system in which we can express \( \mathbf{x} \).

A dependent system is ‘redundant’

Let \( \mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \). Is \( \{ \mathbf{v}_1 \} \) linearly independent? [here: \( p = 1 \)]

A system consisting of a nonzero vector [at least one nonzero entry] is always linearly independent: True - False?

Are the following systems linearly independent:

\[
\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}, \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -10 \\ 0 \end{bmatrix} \right\}, \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \end{bmatrix} \right\}?
\]
A system \( \{u, v\} \) is linearly dependent when __________?

Let \( v_1 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}; \quad v_2 = \begin{bmatrix} 4 \\ 1 \\ 5 \end{bmatrix}; \quad v_3 = \begin{bmatrix} -2 \\ 3 \\ 1 \end{bmatrix}; \)

(a) Determine if \( \{v_1, v_2, v_3\} \) is linearly independent

(b) If possible find a linear dependence relation among \( v_1, v_2, v_3 \).

**Solution:** We must determine if the system:

\[
x_1 \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} + x_2 \begin{bmatrix} 4 \\ 1 \\ 5 \end{bmatrix} + x_3 \begin{bmatrix} -2 \\ 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}
\]

has a nontrivial solution (Trivial solution: \( x_1 = x_2 = x_3 = 0 \))
Augmented syst:  Echelon 1st step  Echelon 2nd step

\[
\begin{bmatrix}
1 & 4 & -2 & 0 \\
1 & 1 & 3 & 0 \\
2 & 5 & 1 & 0 \\
\end{bmatrix}
\begin{bmatrix}
1 & 4 & -2 & 0 \\
0 & -3 & 5 & 0 \\
0 & -3 & 5 & 0 \\
\end{bmatrix}
\begin{bmatrix}
1 & 4 & -2 & 0 \\
0 & -3 & 5 & 0 \\
0 & 0 & 0 & 0 \\
\end{bmatrix}
\]

- This system is equivalent to original one.
- Variable \( x_3 \) is free.
- Select \( x_3 = 3 \) (to avoid fractions) and back-solve for \( x_2 \) (\( x_2 = 5 \)), and \( x_1 \) (\( x_1 = -14 \))
- Conclusion: there is a nontrivial solution
- NOT independent

(b) Linear dependence relation: From above,

\[-14v_1 + 5v_2 + v_3 = 0\]
Note: Text uses the reduced echelon form instead of back-solving [Result is clearly the same. Both solutions are OK]

With the reduced row echelon form

\[
\begin{bmatrix}
1 & 0 & 14/3 & 0 \\
0 & 1 & -5/3 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}
\]

\[x_1 = -(14/3)x_3; \quad x_2 = (5/3)x_3\]

select \(x_3 = 3\) then \(x_2 = 5, x_1 = 14\)

Recall: \(x_1, x_2\) are basic variables, and \(x_3\) is free