THE LU FACTORIZATION  [2.5]
LU factorization: Motivation

Suppose we have to solve many linear systems

\[ Ax = b^{(1)}, \quad Ax = b^{(2)}, \quad \ldots, \quad Ax = b^{(p)} \]

where matrix \( A \) is the same - but the right-hand sides are different

Can solve each of them by Gaussian Elimination separately \( \rightarrow \) inefficient

Can get the inverse \( A^{-1} \) then each solution is of the form \( x^{(k)} = A^{-1}b^{(k)} \)

Cost? [Using method based on rref seen in Lec. Notes 8]
There is a 3rd option (Best): Exploit “LU factorization of $A$”

Main result is this:

*Gaussian elimination algorithm can provide as a by-product a *factorization* of $A$ into the product of a lower triangular matrix $L$ with ones on the diagonal, and an upper triangular matrix $U$:

$$A = LU$$

In addition:

*This factorization is obtained at virtually no extra cost.*

How would you solve systems with multiple right-hand sides using this? What does this approach cost?

Next: The LU factorization. Where does it come from and how to get it?
LU factorization – Revisiting GE

We now ignore the right-hand side in GE

Recall: Gaussian elimination amounts to performing \( n - 1 \) successive Gaussian transformations, i.e., multiplications (to the left) by elementary matrices that have ones on the diagonal and the negatives of the multipliers in the column being eliminated.

Set \( A_0 \equiv A \). Then – results of the \( n - 1 \) steps:

\[
A_1 = E_1 A_0 \\
A_2 = E_2 A_1 = E_2 E_1 A_0 \\
A_3 = E_3 A_2 = E_3 E_2 E_1 A_0 \\
\cdots = \cdots \\
A_{n-1} = E_{n-1} E_{n-2} \cdots E_2 E_1 A_0
\]
$A_{n-1} \equiv U$ is an upper triangular matrix.

We have $U = E_{n-1}E_{n-2}\cdots E_2E_1A$ or:

$$A = \underbrace{E_1^{-1}E_2^{-1}E_3^{-1}\cdots E_{n-1}^{-1}}_{L} U \equiv LU$$

$E_1, E_2, \cdots, E_{n-1}$ are all lower triangular matrices with ones on the diagonal.

What is the inverse of a matrix $E_j$?

Each $E_j^{-1}$ is lower triangular with ones on the diagonal.

Show that the product of unit lower triangular matrices is unit lower triangular.

$L = E_1^{-1}E_2^{-1}E_3^{-1}\cdots E_{n-1}^{-1}$ is lower triangular

$L$ has ones on the diagonal.
In the end:

\[ A = LU \]
\[ L = E_1^{-1} E_2^{-1} E_3^{-1} \cdots E_{n-1}^{-1} \]
\[ U = A_{n-1} \]

Called the LU decomposition (or factorization) of \( A \).

**Notes:**

- \( L \) is **Lower triangular**, and has ones on the diagonal – We say that it is *unit lower triangular*

- \( U \) is the last matrix into which \( A \) is transformed from Gaussian elimination. It is *upper triangular*.

- We know how to get \( U \) [last matrix in GE]

- The main issue now is: How can we get \( L \)?
How do we get $L$?

» Could we use: 

$$L = E_1^{-1}E_2^{-1}E_3^{-1} \cdots E_{n-1}^{-1}$$

? Too complex!

» There is a simpler way:

**Theorem.** Assume that Gaussian elimination can terminate (no division by zero) and let $U$ be the final triangular matrix obtained and $L$ the lower triangular matrix with $l_{ii} = 1$, and, for $i > k$, $l_{ik} = \text{piv}_{ik}$, the multiplier used to eliminate row $i$ in step $k$. Then: $A = LU$.

» $l_{kk} = 1$ and for $i \neq k$, $l_{ik} = $ multiplier $a_{ik}/a_{kk}$ at $k$-th step of GE.

» The matrix $A$ is the product of a unit lower triangular matrix $L$ and an upper triangular matrix $U$. 

Text: 2.5 – LU
**LU factorization - an example**

**Example:** Let \( A = \begin{bmatrix} 4 & -2 & 2 \\ -2 & 5 & 3 \\ 2 & 3 & 9 \end{bmatrix} \)

**Step 1** of GE uses the multipliers \( l_{21} = -\frac{1}{2}, \ l_{31} = \frac{1}{2} \).

What is the matrix \( E_1 \) in this case?

Resulting matrix: \( A_1 = \begin{bmatrix} 4 & -2 & 2 \\ 0 & 4 & 4 \\ 0 & 4 & 8 \end{bmatrix} \)

**Step 2** of Gaussian Elimination uses the multiplier \( l_{32} = 1 \).

What is the matrix \( E_2 \)?
Resulting matrix \[ A_2 = \begin{bmatrix} 4 & -2 & 2 \\ 0 & 4 & 4 \\ 0 & 0 & 4 \end{bmatrix} \equiv U \]

Thus: \[ L = \begin{bmatrix} 1 & 0 & 0 \\ -1/2 & 1 & 0 \\ 1/2 & 1 & 1 \end{bmatrix}, \quad U = \begin{bmatrix} 4 & -2 & 2 \\ 0 & 4 & 4 \\ 0 & 0 & 4 \end{bmatrix} \]

Verify that \( A = LU \)

LU factorization of the matrix \( A = \begin{pmatrix} 2 & 4 & 6 \\ 1 & 5 & 9 \\ 1 & 0 & -12 \end{pmatrix} \)

For the same \( A \) compute the 3rd column of \( A^{-1} \).
How would you compute the inverse of a matrix given its LU factorization?

Show how to use the LU factorization to solve linear systems with the same matrix $A$ and different right-hand sides $b$.

True or false: “Computing the LU factorization of a matrix $A$ involves more arithmetic operations than solving a linear system $Ax = b$ by Gaussian elimination”? 