LU factorization: Motivation

- Suppose we have to solve many linear systems
  \[ Ax = b^{(1)}, \ Ax = b^{(2)}, \ldots, \ Ax = b^{(p)} \]

  where matrix \( A \) is the same - but the right-hand sides are different.

- Can solve each of them by Gaussian Elimination separately \( \rightarrow \) inefficient.

- Cost?

- Can get the inverse \( A^{-1} \) then each solution is of the form \( x^{(k)} = A^{-1}b^{(k)} \).

- Cost? [Using method based on rref seen in Lec. Notes 8]

LU factorization – Revisiting GE

- We now ignore the right-hand side in GE.

  Recall: Gaussian elimination amounts to performing \( n - 1 \) successive Gaussian transformations, i.e., multiplications (to the left) by elementary matrices that have ones on the diagonal and the negatives of the multipliers in the column being eliminated.

- Set \( A_0 \equiv A \). Then – results of the \( n - 1 \) steps:

\[
\begin{align*}
A_1 &= E_1A_0 \\
A_2 &= E_2A_1 = E_2E_1A_0 \\
A_3 &= E_3A_2 = E_3E_2E_1A_0 \\
& \quad \vdots \\
A_{n-1} &= E_{n-1}E_{n-2} \cdots E_2E_1A_0
\end{align*}
\]

- How would you solve systems with multiple right-hand sides using this? What does this approach cost?

- Next: The LU factorization. Where does it come from and how to get it?
An upper triangular matrix $A_{n-1} \equiv U$ is an upper triangular matrix.

We have $U = E_{n-1}E_{n-2} \cdots E_2E_1A$ or:

$$A = E_1^{-1}E_2^{-1}E_3^{-1} \cdots E_{n-1}^{-1}U \equiv LU$$

$E_1, E_2, \ldots, E_{n-1}$ are all lower triangular matrices with ones on the diagonal.

What is the inverse of a matrix $E_j$?

Each $E_j^{-1}$ is lower triangular with ones on the diagonal.

Show that the product of unit lower triangular matrices is unit lower triangular.

$L = E_{n-1}^{-1}E_{n-2}^{-1}E_{n-3}^{-1} \cdots E_1^{-1}$ is lower triangular.

$L$ has ones on the diagonal.

In the end:

$$A = LU$$

with:

$$L = E_1^{-1}E_2^{-1}E_3^{-1} \cdots E_{n-1}^{-1}$$

$$U = A_{n-1}$$

Called the LU decomposition (or factorization) of $A$.

Notes:

$L$ is Lower triangular, and has ones on the diagonal – We say that it is unit lower triangular.

$U$ is the last matrix into which $A$ is transformed from Gaussian elimination. It is upper triangular.

We know how to get $U$ [last matrix in GE]

The main issue now is: How can we get $L$?

How do we get $L$?

Could we use: $L = E_{n-1}^{-1}E_{n-2}^{-1}E_{n-3}^{-1} \cdots E_1^{-1}$? Too complex!

There is a simpler way:

**Theorem.** Assume that Gaussian elimination can terminate (no division by zero) and let $U$ be the final triangular matrix obtained and $L$ the lower triangular matrix with $l_{ii} = 1$, and, for $i > k$, $l_{ik} = \pi iv_{ik}$, the multiplier used to eliminate row $i$ in step $k$. Then:

$$A = LU.$$ 

$l_{kk} = 1$ and for $i \neq k$, $l_{ik} = \text{multiplier } a_{ik}/a_{kk}$ at $k$-th step of GE.

The matrix $A$ is the product of a unit lower triangular matrix $L$ and an upper triangular matrix $U$.

LU factorization - an example

Example: Let $A = \begin{bmatrix} 4 & -2 & 2 \\ -2 & 5 & 3 \\ 2 & 3 & 9 \end{bmatrix}$

**Step 1** of GE uses the multipliers $l_{21} = -1/2$, $l_{31} = 1/2$.

What is the matrix $E_1$ in this case?

Resulting matrix: $A_1 = \begin{bmatrix} 4 & -2 & 2 \\ 0 & 4 & 4 \\ 0 & 4 & 8 \end{bmatrix}$

**Step 2** of Gaussian Elimination uses the multiplier $l_{32} = 1$.

What is the matrix $E_2$?
Resulting matrix

\[
A_2 = \begin{bmatrix}
4 & -2 & 2 \\
0 & 4 & 4 \\
0 & 0 & 4
\end{bmatrix} \equiv U
\]

Thus:

\[
L = \begin{bmatrix}
1 & 0 & 0 \\
-1/2 & 1 & 0 \\
1/2 & 1 & 1
\end{bmatrix}
\quad U = \begin{bmatrix}
4 & -2 & 2 \\
0 & 4 & 4 \\
0 & 0 & 4
\end{bmatrix}
\]

Verify that \( A = LU \)

LU factorization of the matrix

\[
A = \begin{pmatrix}
2 & 4 & 6 \\
1 & 5 & 9 \\
1 & 0 & -12
\end{pmatrix}
\]

For the same \( A \) compute the 3rd column of \( A^{-1} \).