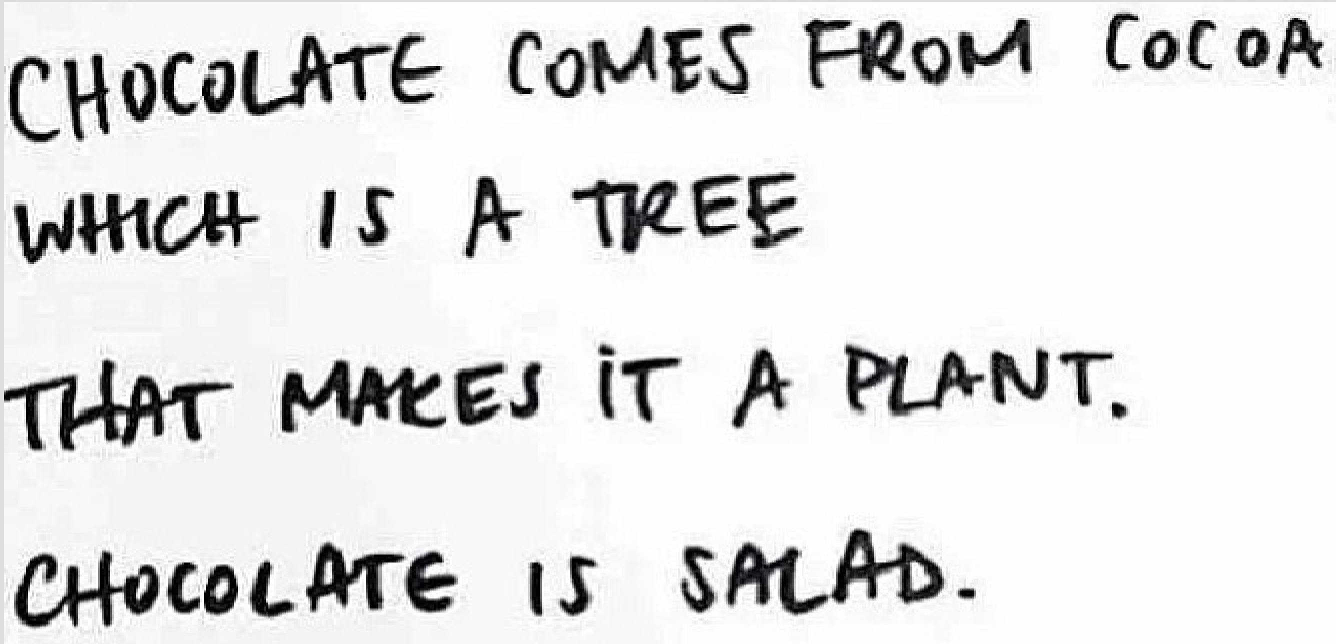
Resolution in FO logic (Ch. 9)



Review: CNF form

Conjunctive normal form is a number of clauses stuck together with ANDs

Each clause can only contain ORs, and logical negation must appears right next to literals

For example: CNF with 3 clauses

$$(A) \land (A \lor \neg B) \land (\neg A \lor \neg B \lor C \lor D)$$

clauses

First-order logic resolution

To do first-order logic resolution we again need to get all the sentences to CNF

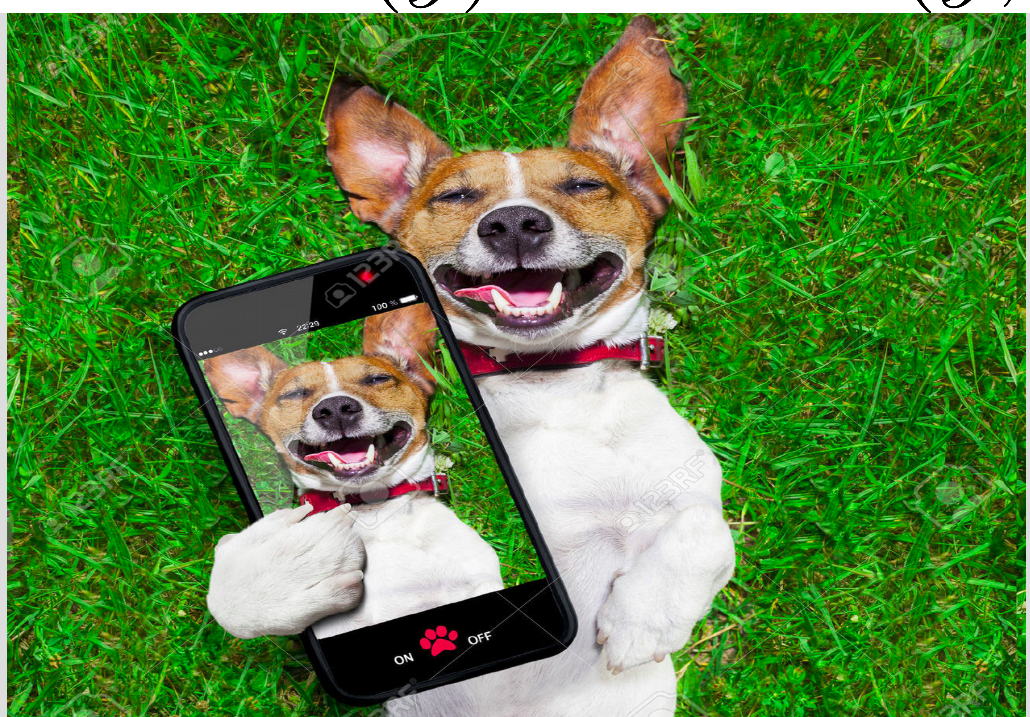
This requires a few more steps for FOL (red):

1. Use logical equivalence to remove implies
2. Move logical negation next to relations
3. Standardize variables
4. Generalize existential quantifiers
5. Drop universal quantifiers
6. Distribute ORs over ANDs

First-order logic resolution

“All dogs that are able to make everyone laugh are owned by someone”

$$\forall x \ [Dog(x) \wedge \forall y \ Person(y) \wedge Laugh(y,x)] \\ \Rightarrow [\exists y \ Person(y) \wedge Owns(y,x)]$$



First-order logic resolution

$$\forall x\ [Dog(x) \land \forall y\ Person(y) \land Laugh(y,x)]$$
$$\Rightarrow [\exists y\ Person(y) \land Owns(y,x)]$$

I have avoided putting quantifiers anywhere except the left for simplicity (as you will see)

There is always a equivalent form with all quantifiers to the left of the main sentence

But the above sentence is logically valid

1. convert implies

As CNF only has ORs and ANDs, we use this:

$$A \Rightarrow B \equiv \neg A \vee B$$

If there is a $\iff$, we use the following first:

$$A \iff B \equiv A \Rightarrow B \wedge B \Rightarrow A$$

First-order logic only allows these logical ops:
$\neg, \vee, \wedge, \Rightarrow, \iff$

So we will have reduced everything to just negation, ANDs and ORs

1. convert implies

$$\forall x \ [Dog(x) \land \forall y \ Person(y) \land Laugh(y,x)] \\ \Rightarrow [\exists y \ Person(y) \land Owns(y,x)]$$

... converts to...

$$\forall x \ \neg[Dog(x) \land \forall y \ Person(y) \land Laugh(y,x)] \\ \lor[\exists y \ Person(y) \land Owns(y,x)]$$

This is now the statement:

"Dogs are either not thought as funny by everyone or owned by someone"

2. move negation to right

Putting negation next to relationships requires two things:

1. De Morgan's laws:

$$\neg(A \vee B) \equiv (\neg A \wedge \neg B)$$
$$\neg(A \wedge B) \equiv (\neg A \vee \neg B)$$

2. Quantifier negation:

$$\neg(\forall x \;\; A(x)) \equiv (\exists x \;\; \neg A(x))$$
$$\neg(\exists x \;\; A(x)) \equiv (\forall x \;\; \neg A(x))$$

2. move negation to right

$$\forall x\ \neg[Dog(x) \wedge \forall y\ Person(y) \wedge Laugh(y,x)]$$
$$\vee[\exists y\ Person(y) \wedge Owns(y,x)]$$

... converts to...

$$\forall x\ [\neg Dog(x) \vee \exists y\ \neg Person(y) \vee \neg Laugh(y,x)]$$
$$\vee[\exists y\ Person(y) \wedge Owns(y,x)]$$

This is now the statement:
“All things are either (not a dog or not funny to a human or funny to a non-human) or owned by someone”

3. standardize variables

It is possible to reuse the same variable in multiple parts of a sentence, such as y in:

$$\forall x\ (\exists y\ A(x, y)) \Rightarrow (\exists y\ B(x, y))$$

You can just rename a variable to make it clear that there is no conflict (having quantifiers on the left ensures there is no confusion)

rename this y to z

$$\forall x\ (\exists y\ A(x, y)) \Rightarrow (\exists y\ B(x, y))$$
$$\equiv \forall x \exists y, z\ A(x, y) \Rightarrow B(x, z)$$

3. standardize variables

$$\forall x\ [\neg Dog(x) \lor \exists y\ \neg Person(y) \lor \neg Laugh(y, x)]$$
$$\lor [\exists y\ Person(y) \land Owns(y, x)]$$

... converts to...

$$\forall x, \exists y, z\ [\neg Dog(x) \lor \neg Person(y) \lor \neg Laugh(y, x)]$$
$$\lor [Person(z) \land Owns(z, x)]$$

The meaning is still the same as last time, but might be easier to understand in half-English: “Every x is either (not a dog, not funny to y or y is not a person) or (person z owns x)”

4. generalize existential

We have talked before about how to make a new object for an existential quantifier:

$\text{Objects} = \{\text{a, b, c}\}$ $\rightarrow$ $\text{Objects} = \{\text{a, b, c, A1}\}$

$\exists x \ A(x)$ $\qquad$ A(A1)

However, the situation is more difficult for existential inside universal quantifier:

$\text{Objects} = \{\text{a, b, c}\}$ $\rightarrow$ $\text{Objects} = \{\text{a, b, c, A1}\}$

$\forall x \exists y \ A(x, y)$?? $\forall x \ A(x, A1)$

Does this work?

4. generalize existential

This does not work...

$$\text{Objects} = \{\text{a, b, c, A1}\} \quad \forall x\ A(x, A1) \rightarrow \text{Objects} = \{\text{a, b, c}\} \quad \exists y \forall x\ A(x, y)$$

This is saying there is a single object (A1), which is true for all x

To properly represent existential on the inside, we need to use a function of x to represent y:

$$\text{Objects} = \{\text{a, b, c}\} \quad \forall x \exists y\ A(x, y) \rightarrow \text{Objects} = \{\text{a, b, c}\} \quad \forall x\ A(x, F(x))$$

4. generalize existential

Function review:

Unary relations = Person(x) (is a relation)

Function = child(x) (is an object)

(functions can also have more than one input)

$$\text{Objects} = \{\text{a, b, c}\} \qquad\qquad \text{Objects} = \{\text{a, b, c}\}$$
$$\forall x \exists y \; A(x, y) \quad \rightarrow \quad \forall x \; A(x, F(x))$$

Here the function F(x) is the y for which A(x,y) is true for any given x

(this is called Skolemization)

4. generalize existential

$$\forall x, \exists y, z \ [\neg Dog(x) \vee \neg Person(y) \vee \neg Laugh(y, x)]$$
$$\vee [Person(z) \wedge Owns(z, x)]$$

... converts to...

$$\forall x \ [\neg Dog(x) \vee \neg Person(Y(x)) \vee \neg Laugh(Y(x), x)]$$
$$\vee [Person(Z(x)) \wedge Owns(Z(x), x)]$$

... I give up translating

If there were multiple universal quantifiers, all the variables would be in the function:

$$\forall x, y \exists a \forall z \ A(x, y, z, a) \equiv \forall x, y, z \ A(x, y, z, F(x, y))$$

5. drop universal quantifiers

As we got rid of existential, there is no confusion about the quantifiers...

So we just simply drop the “for all”s:

$$\forall x\ [\neg Dog(x) \vee \neg Person(Y(x)) \vee \neg Laugh(Y(x), x)] \\ \vee [Person(Z(x)) \wedge Owns(Z(x), x)]$$

... converts to...

$$[\neg Dog(x) \vee \neg Person(Y(x)) \vee \neg Laugh(Y(x), x)] \\ \vee [Person(Z(x)) \wedge Owns(Z(x), x)]$$

6. distribute AND/OR

To get in CNF form, we need all clauses to only contain ORs, and be separated by ANDs:

$$A \vee (B \wedge C) \equiv (A \vee B) \wedge (A \vee C)$$

(basic logic rules of equivalence)

A	B	C	B^C	AV(B^C)	AVB	AVC	(AVB)^(AVC)
T	T	T	T	T	T	T	T
T	T	F	F	T	T	T	T
T	F	T	F	T	T	T	T
T	F	F	F	T	T	T	T
F	T	T	T	T	T	T	T
F	T	F	F	F	T	F	F
F	F	T	F	F	F	T	F
F	F	F	F	F	F	F	F

6. distribute AND/OR

$$A = [\neg Dog(x) \vee \neg Person(Y(x)) \vee \neg Laugh(Y(x), x)]$$

$$B = Person(Z(x))$$

$$C = Owns(Z(x), x)$$

- Substitute into:

$$A \vee (B \wedge C) \equiv (A \vee B) \wedge (A \vee C)$$

$$[\neg Dog(x) \vee \neg Person(Y(x)) \vee \neg Laugh(Y(x), x)] \vee [Person(Z(x)) \wedge Owns(Z(x), x)]$$

... converts to...

$$\left(\neg Dog(x) \vee \neg Person(Y(x)) \vee \neg Laugh(Y(x), x) \vee Person(Z(x))\right) \wedge \left(\neg Dog(x) \vee \neg Person(Y(x)) \vee \neg Laugh(Y(x), x) \vee Owns(Z(x), x)\right)$$

Resolution in FO logic

Once you have the first-order logic in CNF-ish form, resolution is almost the same

The only difference is that you must unify/substitute any variables that you merge

For example:

$$\big(A(x, y, Z(x)) \lor \neg B(y)\big) \land \big(C(x) \lor B(Y(x))\big)$$

... unify/substitute {y/Y(x)}

$$\big(A(x, Y(x), Z(x)) \lor C(x)\big)$$

Resolution in FO logic

You try it!

1. Use logical equivalence to remove implies
2. Move logical negation next to relations
3. Standardize variables
4. Generalize existential quantifiers
5. Drop universal quantifiers
6. Distribute ORs over ANDs

Convert this to CNF:

$$\forall x\ A(x) \iff \forall y\ B(x, y)$$

Resolution in FO logic

$\forall x \; A(x) \iff \forall y \; B(x,y)$

1. $\left(\forall x \; A(x) \Rightarrow \forall y \; B(x,y)\right) \wedge \left(\forall x \; \forall y \; B(x,y) \Rightarrow A(x)\right)$

1. $\left(\forall x \; \neg A(x) \vee \forall y \; B(x,y)\right) \wedge \left(\forall x \; \boxed{\neg \forall y \; B(x,y)} \vee A(x)\right)$

2. $\left(\forall x \; \neg A(x) \vee \forall y \; B(x,y)\right) \wedge \left(\forall x \; \exists y \; \neg B(x,y) \vee A(x)\right)$

3. (nothing to do)

4. $\left(\forall x \; \neg A(x) \vee \forall y \; B(x,y)\right) \wedge \left(\forall x \; \neg B(x, Y(x)) \vee A(x)\right)$

5. $\left(\neg A(x) \vee B(x,y)\right) \wedge \left(\neg B(x, Y(x)) \vee A(x)\right)$

6. (nothing to do)

The negation goes where show in the blue box, because y is localized to one side, while not x

Resolution in FO logic

Resolution is refutation-complete in first-order logic (due to it being semi-decidable)

So using resolution we can tell if: “a entails b”

But we cannot tell if: “a does not entail b”

Resolution recap:
PL: complete, can do “entails” and “not entail”
FOL: refutation-complete, only does “entails”

Resolution in FO logic

Consider this KB:

$$A(Dog) \vee A(Cat)$$
$$\neg A(Dog)$$
$$\forall x \; A(x) \Rightarrow B(x)$$

If we ask: B(Cat)?

$$A(Dog) \vee A(Cat)$$
$$\neg A(Dog)$$
$$\forall x \; \neg A(x) \vee B(x)$$
$$\neg B(Cat)$$

$$\neg A(Cat)$$

unify {x/Cat}

$$A(Dog)$$

Contradiction!

KB enails B(Cat)

Resolution in FO logic

The last example worked correctly as it identified entailment

However, it has trouble giving us answers to an existential: Ask “exists x, A(x)”?

$A(Dog) \vee A(Cat)$

$\neg A(Dog)$

$\forall x \; \neg A(x) \vee B(x)$

$\forall x \neg A(x)$

unify {x/Cat}

$A(Dog)$

contra-
diction

unify {x/Dog}

This only tells us (2 unify): A(Cat) OR A(Dog)

Resolution in FO logic

Thus, resolution in first-order logic will always tell you if a sentence is entailed

However, it might not be able to tell you for what values it is satisfiable

Similar to the semi-decidable nature of FO logic, resolution is complete if entailment can be found in a finite number of inferences (or “resolves”)

Resolution and equality

Once again, I have avoided equality as it is not much fun to deal with

Two ways to deal with this are:

1. Add rules of equality to KB
2. De/Para-modulation (i.e. more substituting)

Both can increase the complexity of the KB or inference by a large amount, so it is better to just avoid equality if possible

Resolution and equality

There are three basic rules of equality:

1. reflexive: $\forall x \; x = x$
2. symmetric: $\forall x, y \; x = y \Rightarrow y = x$
3. transitive: $\forall x, y, z \; x = y \land y = z \Rightarrow x = z$

Then **for each relation/function** we have to add an explicit statement:

Relations (1 var): $\forall x, y \; x = y \Rightarrow A(x) \iff A(y)$

Functions (2 vars): (= instead of iff)

$$\forall a, b, x, y \; a = x \land b = y \Rightarrow F(a, b) = F(x, y)$$

Resolution and equality

Consider this KB: $A(x) \vee B(x, F(x))$

$\forall x, y \ x = y \Rightarrow B(x, y)$

Would need to be converted into:

$$\forall x \ x = x$$

$$\forall x, y \ x = y \Rightarrow y = x$$

$$\forall x, y, z \ x = y \wedge y = z \Rightarrow x = z$$

$$\forall a, x \ a = x \Rightarrow [A(a) \iff A(x)]$$

$$\forall a, b, x, y \ a = x \wedge b = y \Rightarrow [B(a, b) \iff B(x, y)]$$

$$\forall a, x \ a = x \Rightarrow F(a) = F(x)$$

$$A(x) \vee B(x, F(x))$$

$$\forall x, y \ x = y \Rightarrow B(x, y)$$

Resolution and equality

Consider this KB: $A(x) \vee B(x, F(x))$

$\forall x, y \; x = y \Rightarrow B(x, y)$

Basically, you convert = into a relationship

$\forall x \; Eq(x, x)$

$\forall x, y \; Eq(x, y) \Rightarrow Eq(y, x)$

$\forall x, y, z \; Eq(x, y) \wedge Eq(y, z) \Rightarrow Eq(x, z)$

$\forall a, x \; Eq(a, x) \Rightarrow [A(a) \iff A(x)]$

$\forall a, b, x, y \; Eq(a, x) \wedge Eq(b, y) \Rightarrow [B(a, b) \iff B(x, y)]$

$\forall a, x \; Eq(a, x) \Rightarrow Eq(F(a), F(x))$

$A(x) \vee B(x, F(x))$

$\forall x, y \; Eq(x, y) \Rightarrow B(x, y)$

Resolution and equality

The second option doubles the available inferences instead of doubling the KB

We allow paramodulation, in addition to the normal resolution rule

Paramodulation is essentially substituting with a sentence that contains an equals, while also applying resolution to combine (and ensures there is no conflict in the KB)

Resolution and equality

Consider this KB:

$$A(x) \vee B(F(x, Cat)) \vee C(x, Cat)$$

$$[F(Dog, y) = G(y)] \vee D(y)$$

We can then unify {x/Dog, y/Cat} and get:

$$A(Dog) \vee B(F(Dog, Cat)) \vee C(Dog, Cat)$$

$$[F(Dog, Cat) = G(Cat)] \vee D(Cat)$$

Which we can infer:

$$A(Dog) \vee B(G(Cat)) \vee C(Dog, Cat) \vee D(Cat)$$

1. Like resolution you combine sentences
2. Valid substitutions if necessary

Resolution and equality

Consider this KB:

$$A(x) \lor B(F(x, Cat)) \lor C(x, Cat)$$

$$[F(Dog, y) = G(y)] \lor D(y)$$

We can then unify {x/Dog, y/Cat} and get:

$$A(Dog) \lor B(\boxed{F(Dog, Cat)}) \lor C(Dog, Cat)$$

$$[\boxed{F(Dog, Cat)} = \boxed{G(Cat)}] \lor D(Cat)$$

Which we can infer:

$$A(Dog) \lor B(\boxed{G(Cat)}) \lor C(Dog, Cat) \lor D(Cat)$$

1. Like resolution you combine sentences
2. Valid substitutions if necessary

Resolution efficiency

Four (brief) ways to speed up resolution:

1. Subsumption
2. Unit preference
3. Support set
4. Input resolution

1. and 2. are general and do not effect the completeness of resolution

3. and 4. can limit resolvability

Resolution efficiency

Subsumption is to remove any sentences that are fully expressed by another sentence

Consider this KB: $\forall x \; A(x)$
$A(Cat)$

The first sentence is more general and the second is not adding anything

We could simply reduce the KB to: $\forall x \; A(x)$
(and keep th same meaning)

Resolution efficiency

Unit preference is to always apply a clause containing one literal before any others

Since we want to end up with an empty clause for a contradiction, this will shrink the size of the original clause

one literal

For example: $\big(A(x) \lor B(x) \lor C(x)\big) \land \big(\neg A(x)\big)$

... will resolve to: $\big(B(x) \lor C(x)\big)$

Resolution efficiency

A Support set is artificially restricting the KB and removing (what you think are) irrelevant clauses

The set of clauses you use can be based on the query, so if we have this KB:

$$A(x) \Rightarrow B(x)$$
$$B(x) \Rightarrow C(x)$$
$$\exists x \; A(x)$$

Then we ask: $\exists x$ B(x)?

We can see the middle sentence is worthless, so we can solve it just with the first and third

Resolution efficiency

If the support set contains no equalities, there will be a large efficiency increase

However, if the support set does not contain an important sentence you can reach an incorrect conclusion (about entailment)

Even without equality, eliminating a portion of the KB can give large speed ups (as inference is NP-hard, i.e. exponential)

Resolution efficiency

Input resolution starts with a single sentence, and only tries to apply resolution to that sentence (and the resulting sentences)

The resolution of this earlier example is one:

$A(Dog) \lor A(Cat)$

$\neg A(Dog)$

$\forall x \; \neg A(x) \lor B(x)$

$\boxed{\neg B(Cat)}$

$\boxed{\neg A(Cat)}$

$\boxed{A(Dog)}$

Contradiction!

The blue line is involved in all resolutions