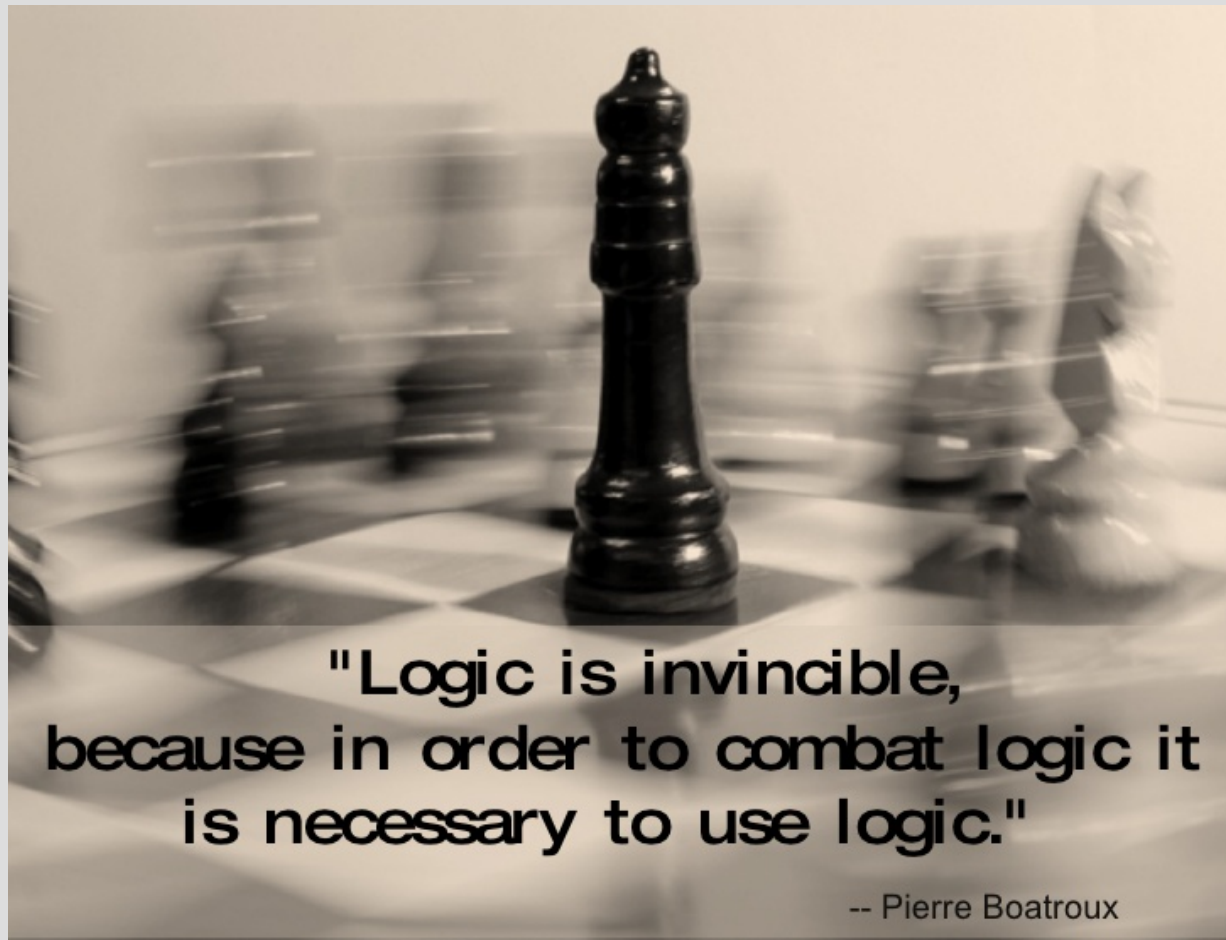


# First order logic (Ch. 8)



**"Logic is invincible,  
because in order to combat logic it  
is necessary to use logic."**

-- Pierre Boatroux

# Review: Propositional logic

Propositional logic builds sentences that relate various symbols with true or false

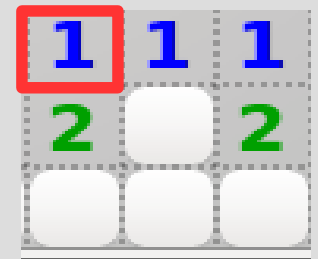
Each symbol is simply a unique identifier, but you cannot “generalize” between them

While this is fairly expressive, it is also quite cumbersome as each part of the environment might need many symbols associated with it

# Review: Propositional logic

For example: to express just the **top left cell** of this mindsweep, we would need to have:

$$\begin{aligned} & P_{1,1,1} \wedge \neg P_{1,1,2} \wedge \neg P_{1,1,3} \\ & \wedge \neg P_{1,1,4} \wedge \neg P_{1,1,5} \wedge \neg P_{1,1,6} \\ & \wedge \neg P_{1,1,7} \wedge \neg P_{1,1,8} \wedge \neg P_{1,1,B} \end{aligned}$$



Sadly in propositional logic we cannot relate these 9 symbols/literals together as “value of cell [1,1]” (and cannot specify this relationship in general for all cells)

# FO logic: definitions

Propositional logic has “propositions” that are either true or false

First order logic (also called “predicate calculus”) has objects and the relation between them is what is important

This can provide a more compact way of expressing the environment (also more complicated since we cannot build truth tables)

# FO logic: definitions

There are two basic things in first order logic:

Also called constant symbols

1. Objects which are some sort of noun or “thing” in the environment (e.g. teacher, bat)
2. Relations among objects, which can be:
  - 2.1. Unary (or properties) which relate to a single object (e.g. red, healthy, boring)
  - 2.2. n-ary which involve more than one
  - 2.3. Functions, one “value” for each input

# FO logic: definitions

Both unary and n-ary relations are similar, just how many variables are involved

Unary and n-ary relations are predicates

Unary and n-ary relations are true/false values (similar to propositional logic)

$President(America, Trump) \wedge President(Mexico, Peña Nieto)$

Functions converts the inputted objects into a single output object (i.e. like coding functions that “return” a **single** object)


$PresidentOf(America) = Trump, PresidentOf(Mexico) = Peña Nieto$

# FO logic: definitions

We can represent any sentence with objects and relations, for example:

“I am sleepy today”  
Object: I, (the “me” of today)  
Relations: Sleepy, Today  
Logic: Sleepy(Today(I))

The set of all objects that we are using are called the domain

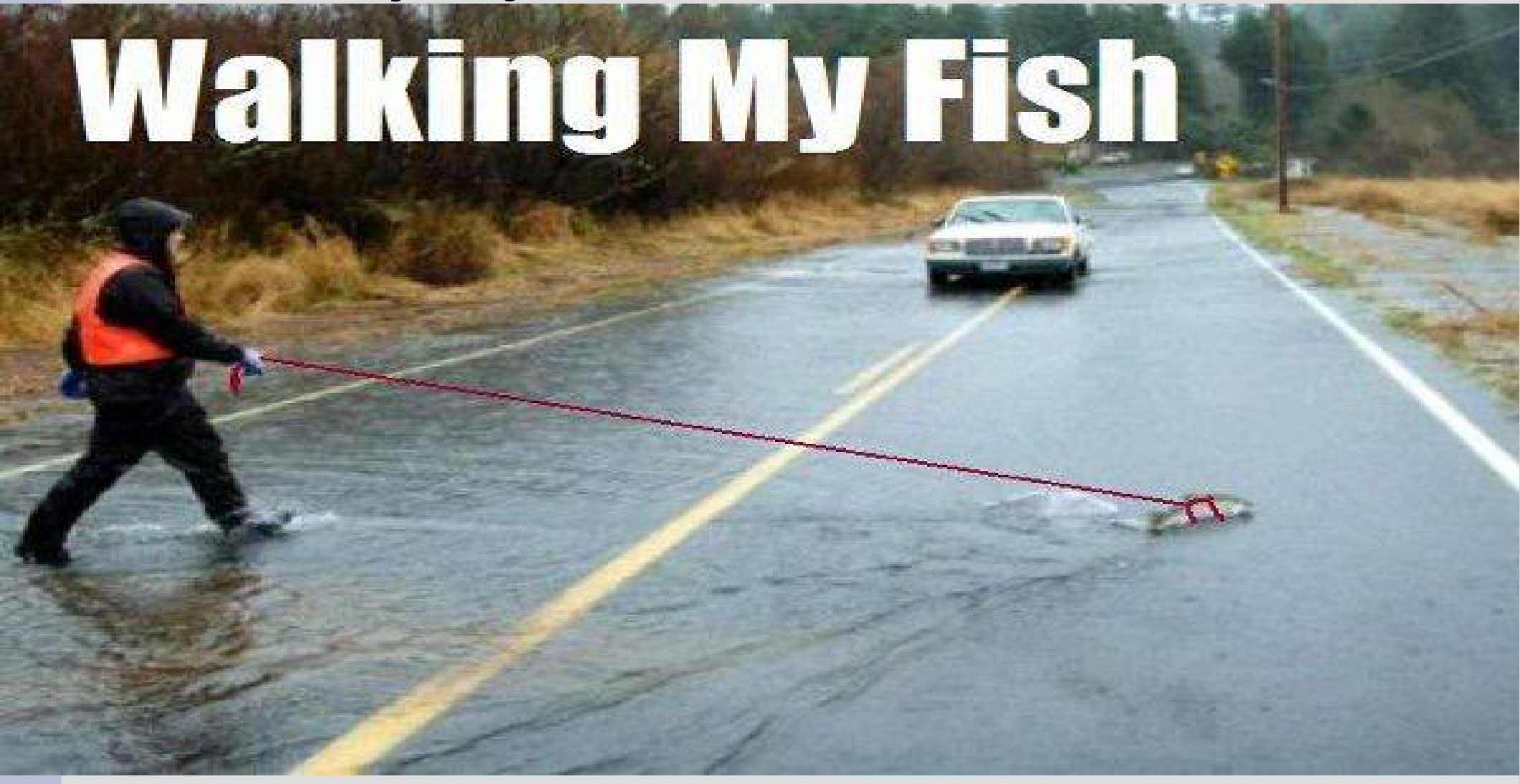


“I howl at full moons”  
Objects: Me, Moon                      Relations: Full, Howl  
Logic: Full(Moon) => Howl(Me)

# FO logic: definitions

Let's identify objects and relations in this:

## Walking My Fish





# FO logic: definitions

Objects:

Person, Car, Road, Fish, Leash

Relations:

Unary: Wet(Fish), Wet(Road), Wet(Car)

n-ary: OnTopOf(Person, Road),

OnTopOf(Car, Road), OnTopOf(Fish, Road)

Functions: attached(Person, Leash) = Fish

# FO logic: definitions

You find objects and relations (what type):



# FO logic: definitions

Objects:

StickPerson, Fish, Pole, Hat, SP'sLeftLeg

Relations examples....

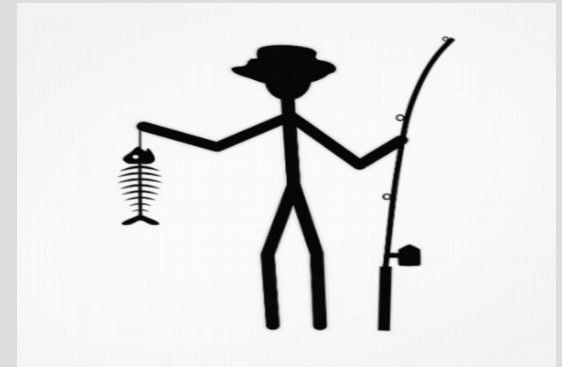
unary: Black(StickPerson)

n-ary: Hold(StickPerson, Fish),

Hold(StickPerson, Pole)

functions:

OnHead(StickPerson), LeftLeg(StickPerson)



# FO logic: definitions

The “arguments” to relations might have an order relation

For example:  $\text{Hold}(\text{StickPerson}, \text{Fish})$  might imply “StickPerson holds Fish”

This is not a symmetric relationship, so  $\text{Hold}(\text{Fish}, \text{StickPerson})$  conveys a different meaning

# FO logic: definitions

Can represent relations as “tuples”  
(generalize “pair” for more than 2 elements)

For example the “Hold” relation might be:  
{<StickPerson, Fish>, <StickPerson, Pole>}

For functions, we normally provide the result:

OnHead:

<StickPerson> → Hat

<Fish> → String

# FO logic: definitions

Side note:

Functions have to be defined for all possible objects in our use of first-order logic

So with the “OnHead” function in the last example, we would also need to define:

$\text{OnHead}(\text{Pole}) = \text{Pole}$

$\text{OnHead}(\text{Hat}) = \text{Hat}$

$\text{OnHead}(\text{SP'sLeftLeg}) = \text{StickPerson} (?)$

# Syntax

Objects and relations form the basis of first order logic, but we also expand our syntax with three things:

1. Quantifiers (existential and universal)
2. Variables (much in the math sense)
3. Equality (as in “=” not “ $\iff$ ” or “ $\equiv$ ”)

Otherwise we have a similar syntax to propositional logic (implies, AND, OR, etc.)

# Existential quantifier

The existential quantifier is  $\exists$ , which means “there exists ...”

For example, if I had a variable “x”, then...

$\exists x \text{ Santa}(x)$

... means “Santa exists” or “Someone is Santa”

$\exists x \text{ InClass}(x) \wedge \text{Hungry}(x)$

... means “Someone in class is hungry” or “At least one person in class is hungry”



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# Variables

A variable is a place-holder for any object

So if we had 3 objects, {Sue, Alex, Devin}, we could formally write:

$$\exists x \textit{Santa}(x)$$

As...

$$\textit{Santa}(\textit{Sue}) \vee \textit{Santa}(\textit{Alex}) \vee \textit{Santa}(\textit{Devin})$$

... or in English: “Someone is Santa”,  
“Santa is Sue, Alex or Devin”

# Universal quantifier

The universal quantifier is denoted by  $\forall$   
means “for all ...”

Thus,  $\forall x \text{ Santa}(x)$  ... means “Everyone is a Santa”

If our objects were again {Sue, Alex, Devin},  
then this would mean:

$\text{Santa}(Sue) \wedge \text{Santa}(Alex) \wedge \text{Santa}(Devin)$

# Quantifier

As  $\exists$  is basically ORs and  $\forall$  is ANDs, we can apply De Morgan's laws:

$$\neg(\exists x \text{ Santa}(x)) \equiv \forall x \neg \text{Santa}(x)$$

In words “No Santa exists” is the same as “Everyone is not Santa” (or “No one is Santa”)

You can have multiple quantifiers as well:

$$\exists x \exists y \text{ SnapChat}(x, y) \equiv \exists x, y \text{ SnapChat}(x, y)$$

This means “Two people are snapchatting”  
(Note: this could also mean snapchatting self)

# Quantifier

The order of quantifiers also matters:

$\forall x \exists y \text{ Mother}(x) = y$  means “For every person  $x$ , they have some mother  $y$ ” or “All people have some mother”

However in the opposite order:

$\exists y \forall x \text{ Mother}(x) = y$  means “There is some person  $y$ , who is the mother to everyone” or “Everyone has the same mother”

# Quantifier

Write these two sentences in logic:

1. “Someone is happy yet sleepy”
2. “Everyone in class is thinking”

# Quantifier

Write these two sentences in logic:

1. “Someone is happy yet sleepy”

$$\exists x \textit{Happy}(x) \wedge \textit{Sleepy}(x)$$

2. “Everyone in class is thinking”

$$\forall x \textit{InClass}(x) \Rightarrow \textit{Thinking}(x)$$

Normally this is the case:

For “ $\exists$ ” you use  $\wedge$

For “ $\forall$ ” you use  $\Rightarrow$

# Equality

In logic, equality means two things are the same (much as it does in math)

For example,  $Sue = Alex$  would imply Sue and Alex are the same people

This is often useful with variables:

$$\forall x, y \neg(x = y) \Rightarrow \neg(Midterm(x) = Midterm(y))$$

... which means “No two (different) people have the same midterm score” (unique scores)



# Assumptions

Being completely expressive in first order logic can be difficult at times

In the last statement you need the “ $\neg(x = y)$ ” (which I will abbreviate often as: “ $x \neq y$ ”) to ensure that the variable does not reference the same person/object

However, in general two objects could be the same thing...

# Assumptions

Try to formally express:

“My only brothers are Bob and Jack”

# Assumptions

Try to formally express:

“My only brothers are Bob and Jack”

$Brother(James, Bob)$

$\wedge Brother(James, Jack)$

$\wedge Jack \neq Bob$

$\wedge \forall x Brother(James, x) \Rightarrow (x = Bob \vee x = Jack)$

This is overly complicated as we have to specify that everyone else is not my brother and that Jack and Bob are different people

# Assumptions

For this reason, we make 3 assumptions:

1. Objects are unique (i.e.  $Bob \neq Jack$  always)
2. All un-said sentences are false  
Thus, if I only say:  $Brother(James, Bob)$   
then I imply:  $\neg Brother(James, Jack)$
3. Only objects I have specified exist  
(i.e. I assume a person  $Davis$  does not exist as I never mentioned them)

# Assumptions

These assumptions make it easier to write sentences more compactly

Under these assumptions, “My sisters are Alice and Grace” can be represented as:

*Sister(James, Alice)  $\wedge$  Sister(James, Grace)*

These assumptions do make it harder to say more general sentences, such as: “Two of my sisters are Alice and Grace”