Assignment #7: Eigenvalues, Eigenvectors, and Diagonalization

Due date: Wednesday, April 3, 2019 (9:15 am)
For full credit you must show all of your work.

1. (4 points) Is \(x = \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}\) an eigenvector of \(A = \begin{bmatrix} 2 & -9 & -6 \\ 0 & 1 & 0 \\ 0 & -2 & -1 \end{bmatrix}\)? Please show all of your work.

2. (4 points) Is \(-1\) an eigenvalue of \(A = \begin{bmatrix} 2 & -9 & -6 \\ 0 & 1 & 0 \\ 0 & -2 & -1 \end{bmatrix}\)? Please show all of your work.

[Hint: check to see if \(-1\) is a root of the characteristic equation of \(A\).]

3. (6 points) Determine the eigenvalues of the following matrices. Hint: use the rules of thumb discussed in class to save effort where possible

\[
A = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 1 & 2 & 3 \end{bmatrix}, \quad C = \begin{bmatrix} 9 & 0 & 0 \\ 8 & 6 & 0 \\ 7 & 5 & 4 \end{bmatrix}
\]

4. (12 points) Determine the eigenvalues and eigenvectors of the following matrices. Hint: use the characteristic equation. Express your answer in terms of \(a\), \(b\), and \(c\).

\[
A = \begin{bmatrix} a & a \\ a & a \end{bmatrix}, \quad B = \begin{bmatrix} a & a \\ b & b \end{bmatrix}, \quad C = \begin{bmatrix} a & b \\ b & a \end{bmatrix}, \quad D = \begin{bmatrix} a & b \\ 0 & c \end{bmatrix}
\]

5. (3 points) If \(A = \begin{bmatrix} 2 & -9 & -6 \\ 0 & 1 & 0 \\ 0 & -2 & -1 \end{bmatrix}\) = \( \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 1 \\ 0 & 1 & -1 \end{bmatrix}\) \( \begin{bmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\) \( \begin{bmatrix} 1 & -5 & -2 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix}\), what are the eigenvalues and eigenvectors of \(A\)?

6. (9 points) Compute the eigenvalues and eigenvectors of \(A = \begin{bmatrix} -1 & -2 & -2 \\ 2 & 3 & 2 \\ 2 & 2 & 3 \end{bmatrix}\). Please show all of your work.

7. (3 points) Let \(A = \begin{bmatrix} -1 & -2 & -2 \\ 2 & 3 & 2 \\ 2 & 2 & 3 \end{bmatrix}\). Find a diagonal matrix \(D\) and an invertible matrix \(P\) such that \(A = PDP^{-1}\).
8. (6 points) Give an example of a $2 \times 2$ matrix that is invertible but not diagonalizable, or explain why such a matrix does not exist. Give an example of a $2 \times 2$ matrix that is diagonalizable but not invertible, or explain why such a matrix does not exist. Try to think of examples that are different from the ones provided in the lecture slides.

9. (17 points) Indicate whether the following statements must be True, or may be False:

a. Every $2\times2$ matrix $A$ is guaranteed to have exactly two distinct eigenvalues $\lambda_1 \neq \lambda_2$.

b. Every $2\times2$ matrix $A$ is guaranteed to have exactly two distinct eigenvectors $v_1 \neq v_2$.

c. Every diagonalizable $2\times2$ matrix $A$ is guaranteed to have exactly two distinct eigenvalues $\lambda_1 \neq \lambda_2$, and two linearly independent eigenvectors $v_1$ and $v_2$.

d. An $n \times n$ matrix can have at most $n$ real eigenvalues.

e. If $\lambda$ is a non-zero eigenvalue of an invertible matrix $A$ then $1/\lambda$ is an eigenvalue of $A^{-1}$.

f. If $v$ is a non-zero eigenvector of an invertible matrix $A$ then $v$ is also an eigenvector of $A^{-1}$.

g. If $v$ is a non-zero eigenvector of a matrix $A$ then $v$ is also an eigenvector of $B = A - \alpha I$ where $\alpha$ is any scalar value.

h. Any square matrix $A$ has the same eigenvalues as its transpose $A^T$.

i. Any square matrix $A$ has the same eigenvectors as its transpose $A^T$.

j. If two matrices $A$ and $B$ are similar, then they will have the same eigenvalues.

k. If two matrices $A$ and $B$ have the same eigenvalues, then they are similar.

l. If two matrices $A$ and $B$ have the same determinant, then they will have the same eigenvalues.

m. If two matrices $A$ and $B$ have the same eigenvalues, then they will have the same determinant.

n. If two matrices $A$ and $B$ have the same eigenvalues, then they will have the same eigenvectors.

o. If $U$ is an echelon form of $A$ then the eigenvalues of $U$ will be the same as the eigenvalues of $A$.

p. If any vector in $\mathbb{R}^n$ can be expressed as a linear combination of the eigenvectors of an $n \times n$ matrix $A$, then $A$ is diagonalizable.

q. If an $n \times n$ matrix $A$ is diagonalizable, then every vector in $\mathbb{R}^n$ will be an eigenvector of $A$.

10. (9 points) Suppose the fixed population of a small state is divided into three groups: urban, suburban, and rural. If the total population is 1,000,000 and, every year, 10% of the rural population moves to the city and 20% of the rural population moves to the suburbs, while 10% of the suburban population moves to the city and 20% of the suburban population moves to the countryside, but only 10% of the population already living in a city leaves the city and all of those go to the suburbs, will the total number of inhabitants in each region eventually reach a steady state, and, if so, what will that number be?
11. (9 points) Use the simple version of the PageRank algorithm presented in class, with a 10% dampening factor, to calculate the relative importance of each website in the following small network. What is the Markov matrix associated with this system? What is the importance vector you compute? List the websites in their order of importance. You should use Matlab to perform these calculations, but for full credit you need to show all of your work, including intermediate steps.

12. (6 points) Use three iterations of the power method to estimate the largest eigenvalue and a corresponding eigenvector of \( A = \begin{bmatrix} 5 & -8 \\ 4 & -7 \end{bmatrix} \), starting with \( x_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \).

You may want to write a script in Matlab to help with the computations. Compare your estimates to the true values. For full credit, you must show all of your work. Please report each of the intermediate estimates as well as the final estimates \( x_3 \) and \( \lambda_3 \).

13. (6 points) Use two iterations of the inverse power method to estimate the smallest eigenvalue and a corresponding eigenvector of \( A = \begin{bmatrix} 5 & -8 \\ 4 & -7 \end{bmatrix} \), starting with \( x_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \).

You may want to write a script in Matlab to help with the computations. Compare your estimates to the true values. For full credit, you must show all of your work. Please report the intermediate estimates as well as the final estimates \( x_2 \) and \( \lambda_2 \).

14. (6 points) Use three iterations of the shifted inverse power method to estimate the middle eigenvalue and a corresponding eigenvector of \( A = \begin{bmatrix} 1 & -1 & 1 \\ -6 & -4 & -9 \\ 4 & 4 & 7 \end{bmatrix} \), starting with \( x_0 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \) and \( \alpha = 1 \).

You may use Matlab to help with the arithmetic. Compare your estimates to the true values.