Assignment #8: Iterative Methods for Computing Eigenvalues, Vector and Matrix Norms, Inner Product, Orthogonality, Orthogonal Sets, Orthogonal Projections

Due date: Monday, April 15, 2019 (9:15am)

Name: ___________________________________________________________
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For full credit you must show all of your work.

1. (16 points) Using the programming language of your choice (Matlab is recommended): write a function that accepts as input an arbitrary \( n \times n \) matrix \( A \) and an initial vector \( x_0 \) in \( \mathbb{R}^n \) and uses the Power Method to obtain and return an estimate of an eigenvector \( x \) associated with the largest eigenvalue \( \lambda \) of \( A \), as well as the largest eigenvalue itself. You are advised to use the simple algorithm presented in class, which iterates the equation \( x_{k+1} = Ax_k / ||Ax_k|| \) to find the eigenvector sought, and then use the Rayleigh quotient: \( x^T Ax / x^T x \) to estimate \( \lambda \). Note that Matlab has a built-in function \( \text{norm()} \) that can be used to compute \( ||Ax_k|| \), as well as a built-in function \( \text{transpose()} \), though the transpose of a matrix or vector can also be obtained in Matlab by appending the character ‘.’

Your program should compute and save each of the intermediate values of \( \lambda \) and \( x \) so that they can be reported. Your program should test for convergence (is the distance between subsequent eigenvector estimates approaching zero? [Hint: remember that any non-zero multiple of any eigenvector is also an eigenvector, so multiplication by \(-1\) is not significant] ). Your program should be robust to situations in which convergence fails to occur (i.e. your program should terminate gracefully with a suitable error message for the user, and should not get stuck in an infinite loop).

Test your program on these matrices:

\[
\begin{bmatrix}
2 & 4 \\
1 & 5
\end{bmatrix}, \begin{bmatrix}
-2 & -1 \\
-1 & -2
\end{bmatrix} \text{ with initial estimate } x_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.
\]

\[
\begin{bmatrix}
1 & 1 \\
0 & 1
\end{bmatrix}, \text{ with initial estimates } x_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, x_0 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \text{ and } x_0 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.
\]

For each matrix, report the number of iterations required to converge within a tolerance of \( \epsilon = 10^{-15} \), as well as the final eigenvector and eigenvalue found. Compare the answers you get to the exact answers returned by Matlab’s \( \text{eig()} \) function.

What happens when you run your program on \( \begin{bmatrix} 5 & -4 \\ -1 & 2 \end{bmatrix} \) with initial estimate \( x_0 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \)? Explain.

Compare the answers you get to the exact answers returned by Matlab’s \( \text{eig()} \) function.

2. Let \( u = \begin{bmatrix} -5 \\ 1 \\ 5 \end{bmatrix} \) and \( v = \begin{bmatrix} -7 \\ 5 \\ 0 \end{bmatrix} \)

   a. (3 points) Compute the inner product \( u \cdot v \)
   b. (4 points) Compute the distance from \( u \) to \( v \)
3. Let \( \mathbf{u} = \begin{bmatrix} -5 \\ 1 \\ 5 \\ -7 \end{bmatrix} \)

a. (7 points) Compute the \( \ell_1 \), \( \ell_2 \), and \( \ell_\infty \) norms of \( \mathbf{u} \)

b. (3 points) Normalize \( \mathbf{u} \) so that it has unit length

4. (5 points) Let \( \mathbf{u} = \begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix} \), \( \mathbf{v} = \begin{bmatrix} 0 \\ 3 \\ 4 \end{bmatrix} \). Compute the correlation between \( \mathbf{u} \) and \( \mathbf{v} \), and between \( 2\mathbf{u} \) and \( \mathbf{v} \).

5. Let \( \mathbf{A} = \begin{bmatrix} 8 & 3 \\ 4 & 6 \\ -8 & 6 \end{bmatrix} \).

a. (2 points) Compute the 1-norm of \( \mathbf{A} \) (denoted \( \| \mathbf{A} \|_1 \))

b. (2 points) Compute the \( \infty \)-norm of \( \mathbf{A} \) (denoted \( \| \mathbf{A} \|_\infty \))

c. (3 points) Compute the Frobenius norm of \( \mathbf{A} \) (denoted \( \| \mathbf{A} \|_F \))

d. (3 points) Compute the spectral norm of \( \mathbf{A} \) (denoted \( \| \mathbf{A} \|_2 \))

6. Consider \( \mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix} \), \( \mathbf{v}_2 = \begin{bmatrix} 1 \\ -3 \\ -1 \end{bmatrix} \), and \( \mathbf{v}_3 = \begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix} \).

a. (3 points) Show that \( \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3 \) form an orthogonal set.

b. (6 points) Express \( \mathbf{x} = \begin{bmatrix} 6 \\ 8 \\ -2 \\ -8 \end{bmatrix} \) as a linear combination of \( \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\} \) by directly calculating the weights \( c_1, c_2, c_3 \) such that \( \mathbf{x} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 \).

7. Let \( \mathbf{x} = \begin{bmatrix} -1 \\ 1 \\ 5 \end{bmatrix} \) and \( \mathbf{u} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \).

a. (4 points) What is the orthogonal projection of \( \mathbf{x} \) onto the subspace of \( \mathbb{R}^3 \) spanned by \( \mathbf{u} \)?

b. (3 points) Express \( \mathbf{x} \) as the sum of two vectors, one of which is in the same direction as \( \mathbf{u} \) and one of which is orthogonal to \( \mathbf{u} \).
8. Consider $v_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$, $v_2 = \begin{bmatrix} 1 \\ -3 \\ -1 \\ -1 \end{bmatrix}$, $v_3 = \begin{bmatrix} 3 \\ -1 \\ -1 \\ -1 \end{bmatrix}$, $v_4 = \begin{bmatrix} -1 \\ 1 \\ 1 \\ -3 \end{bmatrix}$, which form an orthogonal basis for $\mathbb{R}^4$

a. (8 points) Find the orthogonal projection of $x = \begin{bmatrix} 6 \\ 8 \\ 4 \\ -2 \end{bmatrix}$ onto each of the 1D subspaces of $\mathbb{R}^4$ spanned by each of the basis vectors $v_i$.

b. (3 points) Express $x = \begin{bmatrix} 6 \\ 8 \\ 4 \\ -2 \end{bmatrix}$ as a linear combination of $\{v_1, v_2, v_3, v_4\}$.

c. (3 points) Find the closest point $\hat{x}$ to $x$ in the subspace $W$ of $\mathbb{R}^4$ that is spanned by $\{v_3, v_4\}$.

d. (2 points) Express $x$ as the sum of two orthogonal vectors, $w$ which is in the 2D subspace $W$ spanned by $\{v_3, v_4\}$ and $u$ which is in the 2D subspace $W^\perp$.

9. Let $v_1 = \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}$, $v_2 = \begin{bmatrix} 2 \\ 4 \\ 4 \end{bmatrix}$ be a basis for a 2D subspace of $\mathbb{R}^3$ and let $V$ be the matrix whose columns are defined by $v_1$ and $v_2$.

a. (3 points) Is $V$ an orthogonal matrix?

b. (2 points) Let $U$ be a matrix with orthonormal columns $u_1$ and $u_2$ that span the same subspace of $\mathbb{R}^3$ as is spanned by the columns of $V$. Is $U$ an orthogonal matrix?

c. (2 points) Use Matlab to compute $A = UU^T$ and $B = U^TU$. What is the rank of $A$? What is the rank of $B$?

d. (5 points) Compute $UU^T$ where $y = \begin{bmatrix} 6 \\ -6 \\ 3 \end{bmatrix}$. What does the result tell you about $y$?

10. For each of the following, choose the correct option. If the answer is True, explain how you know. If the answer is False, give a counter-example. (8 points)

a. **True** or **False**: every matrix that has orthogonal columns is an orthogonal matrix.

b. **True** or **False**: every linearly independent set of vectors in $\mathbb{R}^2$ is an orthogonal set.
c. True or False: if two non-zero vectors $v_1$ and $v_2$ are orthogonal, then they are also linearly independent.

d. True or False: if $A$ is an orthogonal matrix, then $A$ is invertible.

e. True or False: if $A$ is an orthogonal matrix, then $AA^T = A^TA = I$.

f. True or False: if $A$ is an orthogonal matrix, then $A^T = A^{-1}$.

g. True or False: if $A$ is a matrix that has orthogonal columns, then $A^TA = I$.

h. True or False: if $A$ is a matrix that has orthogonal columns, then $A^TA = D$ where $D$ is diagonal.