Assignment #10: Diagonalization of Symmetric Matrices, Quadratic Forms, Optimization, Singular Value Decomposition

Due date: Monday, May 6, 2019 (9:15am)

Name: ________________________________________________________________

Section Number
Assignment #10: Diagonalization of Symmetric Matrices, Quadratic Forms, Optimization, Singular Value Decomposition

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For full credit you must show all of your work.

1. (10 points) Orthogonally diagonalize  \( A = \begin{bmatrix} 1 & -2 & 4 \\ -2 & -2 & -2 \\ 4 & -2 & 1 \end{bmatrix} \)

2. (8 points) Suppose the symmetric matrix  \( A = \begin{bmatrix} 5 & -2 & 0 \\ -2 & 6 & 2 \\ 0 & 2 & 7 \end{bmatrix} \) can be orthogonally diagonalized as:

\[
A = \begin{bmatrix} -1/3 & 2/3 & 2/3 \\ 2/3 & -1/3 & 2/3 \\ 2/3 & 2/3 & -1/3 \end{bmatrix} \begin{bmatrix} 9 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} -1/3 & 2/3 & 2/3 \\ 2/3 & -1/3 & 2/3 \\ 2/3 & 2/3 & -1/3 \end{bmatrix}
\]

a. Express  \( A \) as the weighted sum of three rank 1 matrices:  \( A = \lambda_1 v_1 v_1^T + \lambda_2 v_2 v_2^T + \lambda_3 v_3 v_3^T \)
where the weights  \( \lambda_1, \lambda_2, \lambda_3 \) are the eigenvalues of  \( A \) and the rank 1 matrices are formed from the outer products of the corresponding orthogonal unit length eigenvectors of  \( A \).

b. Use Matlab to compute  \( ||v_1||_2, ||v_1 v_1^T||_2, ||\lambda_1 v_1 v_1^T||_2, \) and  \( ||A||_2 \). What do you notice?

3. Let  \( v = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \) be an arbitrary non-zero vector in  \( \mathbb{R}^3 \)

a. (2 points) Compute the inner product  \( v^T v = s \). (Express  \( s \) in terms of the entries  \( v_1, v_2, v_3 \) of  \( v \))

b. (2 points) Compute the outer product  \( vv^T = M \). (Express  \( M \) in terms of the entries  \( v_1, v_2, v_3 \) of  \( v \))

c. (2 points) Explain why it is guaranteed that any  \( n \times n \) matrix  \( M \) obtained as the outer product of any non-zero vector  \( v \) in  \( \mathbb{R}^n \) will be symmetric

d. (2 points) Explain why it is guaranteed that any  \( n \times n \) matrix  \( M \) obtained as the outer product of any non-zero vector  \( v \) in  \( \mathbb{R}^n \) will have rank 1

e. (2 points) What are the eigenvalues  \( \lambda_1, \lambda_2, \ldots, \lambda_n \) of  \( M \)? How do you know? (Please express each  \( \lambda_i \) in terms of the entries  \( v_1, v_2, \ldots, v_n \) of  \( v \) as necessary)

f. (2 points) Explain why it is guaranteed that  \( v \) is an eigenvector of  \( M = vv^T \).
[Hint: to do this, you can explain how you know that  \( Mv = \lambda v \)]

g. (2 points) Explain why it is guaranteed that the matrix  \( M = vv^T \) will project any vector  \( w \) in  \( \mathbb{R}^n \) onto the subspace of  \( \mathbb{R}^n \) spanned by  \( v \). [Hint: explain how you know that  \( M(w) = \alpha v \).]
4. (4 points) Show that for any $m \times n$ matrix $M$, the matrix products $M^T M$ and $MM^T$ are guaranteed to be symmetric.

5. (4 points) What is the quadratic form of the matrix $M = \begin{bmatrix} 0 & 1 & -1 \\ 1 & 2 & 0 \\ -1 & 0 & 3 \end{bmatrix}$?

b. (4 points) What is the matrix of the quadratic form $Q(x) = 3x_1^2 + 2x_2^2 + 8x_1x_2 - 6x_2x_3$?

6. Consider the matrix $M = \begin{bmatrix} 2 & 4 & 0 \\ 4 & 0 & -4 \\ 0 & -4 & -2 \end{bmatrix}$.

a. (8 points) Use a change-of-variable to express the quadratic form $Q(x)$ of $M$ without any cross-product terms. [In other words, find $a$, $b$, $c$, and $y = [y_1 \ y_2 \ y_3]^T$ so that $Q(x) = ay_1^2 + by_2^2 + cy_3^2$.]

b. (2 points) Classify this quadratic form.

7. (4 points) For what values of $a$ will the matrix $M = \begin{bmatrix} a & 1 \\ 1 & -2 \end{bmatrix}$ be positive definite? positive semi-definite? indefinite? negative semi-definite? negative definite?

8. (4 points) It can be shown that for any arbitrary $m \times n$ matrix $A$, the matrix $A^T A$ is guaranteed to be symmetric and positive semi-definite. Under what conditions will the product $A^T A$ of an arbitrary $m \times n$ matrix $A$ fail to be positive definite? (Characterize the matrices $A$ for which this outcome will occur.) Please justify your answer.

9. Consider the symmetric matrix $A = \begin{bmatrix} 2 & -1 & -4 \\ 2 & 2 & 2 \\ -1 & -4 & -1 \end{bmatrix}$ which can be orthogonally diagonalized as:

\[
A = \begin{bmatrix} -1/3 & 2/3 & 2/3 \\ 2/3 & -1/3 & 2/3 \\ 2/3 & 2/3 & -1/3 \end{bmatrix} \begin{bmatrix} -6 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} -1/3 & 2/3 & 2/3 \\ 2/3 & -1/3 & 2/3 \\ 2/3 & 2/3 & -1/3 \end{bmatrix}
\]

a. (1 points) What is the maximum of the quadratic form $Q(x)$ of $A$, over all unit length vectors $x$?

b. (1 points) What is the minimum of the quadratic form $Q(x)$ of $A$, over all unit length vectors $x$?

c. (2 points) For what unit vectors $x$ will $Q(x)$ take its minimum value?

d. (2 points) For what unit vectors $x$ will $Q(x)$ take its maximum value?

e. (3 points) Find a unit vector $x$ at which $Q(x) = 0$. Describe the set of unit vectors $x$ at which $Q(x) = 0$. 

\[ A = \begin{bmatrix} -1/3 & 2/3 & 2/3 \\ 2/3 & -1/3 & 2/3 \\ 2/3 & 2/3 & -1/3 \end{bmatrix} \begin{bmatrix} -6 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} -1/3 & 2/3 & 2/3 \\ 2/3 & -1/3 & 2/3 \\ 2/3 & 2/3 & -1/3 \end{bmatrix} \]
10. (24 points) Working by hand, compute the singular value decomposition of the following matrices. (This will be good practice for the final exam).

\[
A = \begin{bmatrix} 2 & 3 \\ 4 & 6 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 1 & -1 \end{bmatrix} \quad C = \begin{bmatrix} 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix}
\]

11. (5 points) Let \( A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & -2 & 1 \end{bmatrix} \). \( A \) is a linear transformation that maps vectors \( x \) in \( \mathbb{R}^3 \) into vectors \( b \) in \( \mathbb{R}^2 \). Consider the set of all possible vectors \( b = Ax \), where \( x \) is of unit length. What is the longest vector \( b \) in this set, and what unit length vector \( x \) is used to obtain it? You can use Matlab to save time with the computations, but please justify your answer.

12. (10 points extra credit) The singular value decomposition of an \( m \times n \) matrix \( A = U\Sigma V^T \) can be expressed as a sum of \( r \) distinct matrices \( B_1 \ldots B_r \), where \( r = \min(m, n) \) and \( B_i = u_i\sigma_i v_i^T \), where \( u_i \) is the \( i \)th column of \( U \), \( \sigma_i \) is the \( i \)th diagonal element of \( \Sigma \) and \( v_i^T \) is the \( i \)th row of \( V \). If we treat an image as an \( m \times n \) matrix, and decompose it using a singular value decomposition, then the sum of all of the components would be equal to the original image. But what is the result if we take the sum of only the first few components? This Matlab exercise provides an opportunity for you to explore that question. Please follow the steps below:

a. Use the command
   
   ```matlab
   A = imread('testpat1.png');
   ```
   
   to load a test pattern image into a matrix \( A \). \( A \) will have dimensions 512 x 512, and its entries will be 8-bit unsigned integers. Note that Matlab has many other images you could also use.

b. Use the command
   
   ```matlab
   [U, S, V] = svd(double(A));
   ```
   
   to obtain the singular value decomposition of \( A \), where \( A = USV^T \). Note that you have to convert the values of \( A \) from \texttt{int} to \texttt{double} before you can apply Matlab’s \texttt{svd()} function to it.

c. In a loop of increasing values of \( k \), for example \( k = 1 : 2 : 100 \), reconstruct a “reduced” version of \( A \) using only the first \( k \) components of the singular value decomposition. Specifically, you need to form the product \( a = usv^T \) where \( u \) is a 512 x \( k \) matrix containing the first \( k \) columns of \( U \), \( s \) is a \( k \) x \( k \) matrix containing the first \( k \) diagonal values in \( S \), and \( v^T \) is a \( k \) x 512 matrix containing the first \( k \) rows of \( V^T \). You can use a command like: \( u = \texttt{U(:, i)} \) to extract the \( i \)th column from \( U \), and a command like: \( u = \texttt{U(:, i:j)} \) to extract a range of columns between the \( i \)th and the \( j \)th. Be careful to extract the correct portions of each component matrix. In your loop, use the commands \texttt{figure} and \texttt{imshow(uint8(a))} to display the series of reconstructed images. (You need to convert the values of the matrix \( a \) back to 8-bit unsigned integers before they can be displayed as an image.)

d. Report your impressions. How many components do you need to use before the image becomes recognizable? After how many components do you stop seeing much difference in the result?

e. Extract and plot the singular values in matrix \( S \). You can do this using the command:
   
   ```matlab
   plot(1:length(diag(S)), diag(S));
   ```
Do you notice a relationship between the sizes of the singular values and the pace of improvement in the quality of the reduced image?

Please pass in: a printout of your code; your answers to the questions above, and one of the reduced images that you obtained, along with an annotation of the number of components you used to construct it (the value $k$).