Orthogonal projectors and the URV decomposition

- Orthogonal subspaces;
- Orthogonal projectors; Orthogonal decomposition;
- The URV decomposition
- Introduction to the Singular Value Decomposition

Orthogonal projectors and subspaces

Notation: Given a supspace $\mathcal{X}$ of $\mathbb{R}^m$ define

$$\mathcal{X}^\perp = \{y \mid y \perp x, \quad \forall\, x \in \mathcal{X}\}$$

➤ Let $Q = [q_1, \cdots, q_r]$ an orthonormal basis of $\mathcal{X}$

✍ How would you obtain such a basis?

➤ Then define orthogonal projector $P = QQ^T$

Properties

(a) $P^2 = P$ (b) $(I - P)^2 = I - P$

(c) $Ran(P) = \mathcal{X}$ (d) $Null(P) = \mathcal{X}^\perp$

(e) $Ran(I - P) = Null(P) = \mathcal{X}^\perp$

➤ Note that (b) means that $I - P$ is also a projector

Proof. (a), (b) are trivial

(c): Clearly $Ran(P) = \{x |\ x = QQ^T y, y \in \mathbb{R}^m\} \subseteq \mathcal{X}$. Any $x \in \mathcal{X}$ is of the form $x = Qy, y \in \mathbb{R}^m$. Take $Px = QQ^T(Qy) = Qy = x$. Since $x = Px$, $x \in Ran(P)$. So $\mathcal{X} \subseteq Ran(P)$. In the end $\mathcal{X} = Ran(P)$.

(d): $x \in \mathcal{X}^\perp \leftrightarrow (x, y) = 0, \forall y \in \mathcal{X} \leftrightarrow (x, Qz) = 0, \forall z \in \mathbb{R}^r \leftrightarrow (Q^T x, z) = 0, \forall z \in \mathbb{R}^r \leftrightarrow Q^T x = 0 \leftrightarrow QQ^T x = 0 \leftrightarrow Px = 0$.

(e): Need to show inclusion both ways.

- $x \in Null(P) \leftrightarrow Px = 0 \leftrightarrow (I - P)x = x \rightarrow x \in Ran(I - P)$
- $x \in Ran(I - P) \leftrightarrow \exists y \in \mathbb{R}^m | x = (I - P)y \rightarrow Px = P(I - P)y = 0 \rightarrow x \in Null(P)$ ■

Result: Any $x \in \mathbb{R}^m$ can be written in a unique way as

$$x = x_1 + x_2, \quad x_1 \in \mathcal{X}, \quad x_2 \in \mathcal{X}^\perp$$

- Proof: Just set $x_1 = Px, \quad x_2 = (I - P)x$
- Note: $$\mathcal{X} \cap \mathcal{X}^\perp = \{0\}$$
- Therefore: $$\mathbb{R}^m = \mathcal{X} \oplus \mathcal{X}^\perp$$
- Called the *Orthogonal Decomposition*

Orthogonal decomposition

- In other words $\mathbb{R}^m = P\mathbb{R}^m \oplus (I - P)\mathbb{R}^m$ or:

$$\mathbb{R}^m = Ran(P) \oplus Ran(I - P) \text{ or:}$$
$$\mathbb{R}^m = Ran(P) \oplus Null(P) \text{ or:}$$
$$\mathbb{R}^m = Ran(P) \oplus Ran(P)^\perp$$

- Can complete basis $\{q_1, \cdots, q_r\}$ into orthonormal basis of $\mathbb{R}^m$, $q_{r+1}, \cdots, q_m$
- $\{q_{r+1}, \cdots, q_m\}$ = basis of $\mathcal{X}^\perp$. $\rightarrow$ $dim(\mathcal{X}^\perp) = m - r.$

Four fundamental supspaces - URV decomposition

Let $A \in \mathbb{R}^{m\times n}$ and consider $\mathrm{Ran}(A)^\perp$

Property 1: $\mathrm{Ran}(A)^\perp = \boldsymbol{Null}(A^T)$

Proof: $x \in \mathrm{Ran}(A)^\perp$ iff $(Ay, x) = 0$ for all y iff $(y, A^Tx) = 0$ for all y ...

Property 2: $\mathrm{Ran}(A^T) = \boldsymbol{Null}(A)^\perp$

- Take $\mathcal{X} = \mathrm{Ran}(A)$ in orthogonal decomoposition.
- Result:

$$\mathbb{R}^m = \boldsymbol{Ran}(A) \oplus \boldsymbol{Null}(A^T)$$
$$\mathbb{R}^n = \boldsymbol{Ran}(A^T) \oplus \boldsymbol{Null}(A)$$

4 fundamental subspaces

$\boldsymbol{Ran}(A)$ $\boldsymbol{Null}(A^T)$,
$\boldsymbol{Ran}(A^T)$ $\boldsymbol{Null}(A)$

- Express the above with bases for $\mathbb{R}^m$:

$$[\underbrace{u_1, u_2, \cdots, u_r}_{Ran(A)}, \underbrace{u_{r+1}, u_{r+2}, \cdots, u_m}_{Null(A^T)}]$$

and for $\mathbb{R}^n$ $$[\underbrace{v_1, v_2, \cdots, v_r}_{Ran(A^T)}, \underbrace{v_{r+1}, v_{r+2}, \cdots, v_n}_{Null(A)}]$$

- Observe $u_i^T A v_j = 0$ for $i > r$ or $j > r$. Therefore

$$U^T A V = R = \begin{pmatrix} C & 0 \\ 0 & 0 \end{pmatrix}_{m \times n} \quad C \in \mathbb{R}^{r \times r} \quad \longrightarrow$$

$$A = URV^T$$

- General class of URV decompositions

➤ Far from unique.

✍ Show how you can get a decomposition in which C is lower (or upper) triangular, from the above factorization.

➤ Can select decomposition so that R is upper triangular $\longrightarrow$ URV decomposition.

➤ Can select decomposition so that R is lower triangular $\longrightarrow$ ULV decomposition.

➤ SVD $=$ special case of URV where $R =$ diagonal

✍ How can you get the ULV decomposition by using only the Householder QR factorization (possibly with pivoting)? [Hint: you must use Householder twice]