✍1 Consider

$$\boldsymbol{A} = \begin{pmatrix} \mathbf{1} & \mathbf{2} & \mathbf{-4} \\ \mathbf{0} & \mathbf{1} & \mathbf{2} \\ \mathbf{0} & \mathbf{0} & \mathbf{2} \end{pmatrix}$$

Eigenvalues of $\boldsymbol{A}$? their algebraic multiplicities? their geometric multiplicities? Is one a semi-simple eigenvalue?

Solution: The eigenvalues of $\boldsymbol{A}$ are 1, and 2. The algebraic multiplicity of 1 is 2. To get the geometric multiplicity of the eigenvalue $\boldsymbol{\lambda = 1}$ we need to eigenvectors. For this we need to solve:

$$\begin{pmatrix} \mathbf{0} & \mathbf{2} & \mathbf{-4} \\ \mathbf{0} & \mathbf{0} & \mathbf{2} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} \end{pmatrix} \boldsymbol{u} = \mathbf{0}.$$

There is only one solution vector (up to a product by a scalar) namely:

$$\begin{pmatrix} \mathbf{1} \\ \mathbf{0} \\ \mathbf{0} \end{pmatrix}$$

So the geometric multiplicity is one. □

✍2 Same questions if $\boldsymbol{a_{33}}$ is replaced by one.

Solution: The matrix become

$$\boldsymbol{A} = \begin{pmatrix} \mathbf{1} & \mathbf{2} & \mathbf{-4} \\ \mathbf{0} & \mathbf{1} & \mathbf{2} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} \end{pmatrix}$$

and now we have one eigenvalue algebraic multiplicity 3.

To get the geometric multiplicity of the eigenvalue $\boldsymbol{\lambda = 1}$ we need to eigenvectors. For this we need to solve:

$$\begin{pmatrix} \mathbf{0} & \mathbf{2} & \mathbf{-4} \\ \mathbf{0} & \mathbf{0} & \mathbf{2} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{pmatrix} \boldsymbol{u} = \mathbf{0}.$$

we still get a geometric mult. of 1. □

✍3 Same questions if in addition $\boldsymbol{a}_{12}$ is replaced by zero.

Solution: Solution: The matrix become

$$\boldsymbol{A} = \begin{pmatrix} \mathbf{1} & \mathbf{0} & \mathbf{-4} \\ \mathbf{0} & \mathbf{1} & \mathbf{2} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} \end{pmatrix}$$

and we also have one eigenvalue with algebraic multiplicity 3. The geometric multiplicity increases to 2. □

✍4 Show that there is at least one eigenvalue and eigenvector of $\boldsymbol{A}$: $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{\lambda}\boldsymbol{x}$, with $\|\boldsymbol{x}\|_2 = \mathbf{1}$

Solution: This comes from the fact that the equation $\boldsymbol{P_A}(\boldsymbol{\lambda}) = \mathbf{det}(\boldsymbol{A} - \boldsymbol{\lambda}\boldsymbol{I}) = \mathbf{0}$ is a polynomial equation and as such it must have at least one root - a well-known result. □

✍5 There is a unitary transformation $\boldsymbol{P}$ such that $\boldsymbol{Px} = \boldsymbol{e_1}$. How do you define $\boldsymbol{P}$?

Solution: This is just the Householder transform.. See Lecture notes number 10. □

✍6 Show that $\boldsymbol{PAP^H} = \left(\begin{array}{c|c} \boldsymbol{\lambda} & \boldsymbol{**} \\ \hline \boldsymbol{0} & \boldsymbol{A_2} \end{array}\right)$.

Solution: This is equivalent to showing that $\boldsymbol{PAP^He_1} = \boldsymbol{\lambda e_1}$. We have

$$\boldsymbol{PAP^He_1} = \boldsymbol{PAPe_1} = \boldsymbol{P(Ax)} = \boldsymbol{P(\lambda x)} = \boldsymbol{\lambda Px} = \boldsymbol{\lambda e_1}$$

□

✍9 Another proof altogether: use Jordan form of $\boldsymbol{A}$ and QR factorization **Solution:** Jordan form:

$$\boldsymbol{A} = \boldsymbol{XJX^{-1}}$$

Let $\boldsymbol{X} = \boldsymbol{Q}\boldsymbol{R_0}$ then:

$$\boldsymbol{A} = \boldsymbol{Q}\boldsymbol{R}_0\boldsymbol{J}\boldsymbol{R}_0^{-1}\boldsymbol{Q}^H \equiv \boldsymbol{Q}\boldsymbol{R}\boldsymbol{Q}^H \quad \text{with} \quad \boldsymbol{R} = \boldsymbol{R}_0\boldsymbol{J}\boldsymbol{R}_0^{-1}$$

$\square$