✍1 Unitary matrices preserve the 2-norm.

Solution: The proof takes only one line if we use the result $(\boldsymbol{Ax}, \boldsymbol{y}) = (\boldsymbol{x}, \boldsymbol{A}^H \boldsymbol{y})$:

$$\|\boldsymbol{Qx}\|_2^2 = (\boldsymbol{Qx}, \boldsymbol{Qx}) = (\boldsymbol{x}, \boldsymbol{Q}^H \boldsymbol{Qx}) = (\boldsymbol{x}, \boldsymbol{x}) = \|\boldsymbol{x}\|_2^2. \qquad \square$$

✍3 When do we have equality in Cauchy-Schwarz?

Solution: From the proof of Cauchy-Schwarz it can be seen that we have equality when $\boldsymbol{x} = \boldsymbol{\lambda y}$, i.e., when they are colinear.□

✍4 Expand $(\boldsymbol{x} + \boldsymbol{y}, \boldsymbol{x} + \boldsymbol{y})$ – What does Cauchy-Schwarz imply?

Solution: You will see that you can derive the triangle inequality from this expansion and the Cauchy-Schwarz inequality. □.

✍ 5 Second triangle inequality.

Solution: Start by invoking the triangle inequality to write:

$$\|\boldsymbol{x}\| = \|(\boldsymbol{x}-\boldsymbol{y}) + \boldsymbol{y}\| \leq \|\boldsymbol{x}-\boldsymbol{y}\| + \|\boldsymbol{y}\| \rightarrow \|\boldsymbol{x}\| - \|\boldsymbol{y}\| \leq \|\boldsymbol{x}-\boldsymbol{y}\|$$

Next exchange the roles of $\boldsymbol{x}$ and $\boldsymbol{y}$:

$$\|\boldsymbol{y}\| - \|\boldsymbol{x}\| \leq \|\boldsymbol{y}-\boldsymbol{x}\| = \|\boldsymbol{x}-\boldsymbol{y}\|$$

The two inequalities $\|\boldsymbol{x}\| - \|\boldsymbol{y}\| \leq \|\boldsymbol{x}-\boldsymbol{y}\|$ and $\|\boldsymbol{y}\| - \|\boldsymbol{x}\| \leq \|\boldsymbol{x}-\boldsymbol{y}\|$ yield the result since they imply that

$$-\|\boldsymbol{x}-\boldsymbol{y}\| \leq \|\boldsymbol{x}\| - \|\boldsymbol{y}\| \leq \|\boldsymbol{x}-\boldsymbol{y}\|$$

□

✍6 Norms are continuous functions in $\mathbb{R}^n$ (or $\mathbb{C}^n$).

Solution: We need to show that we can make $\|y\|$ arbitrarily close to $\|x\|$ by making y 'close' enough to x, where 'close' is measured in terms of the infinity norm distance $d(x, y) = \|x - y\|_\infty$. Define $u = x - y$ and write u in the canonical basis as $u = \sum_{i=1}^n \delta_i e_i$. Then:

$$\|u\| = \|\sum_{i=1}^n \delta_i e_i\| \le \sum_{i=1}^n |\delta_i| \ \|e_i\| \le \max |\delta_i| \sum_{i=1}^n \|e_i\|$$

Setting $M = \sum_{i=1}^n \|e_i\|$ we get $\boxed{\|u\| \le M \max |\delta_i| = M\|x - y\|_\infty}$

Let ϵ be given and take x, y such that $\|x - y\|_\infty \le \frac{\epsilon}{M}$. Then, by using the second triangle inequality we obtain:

$$| \ \|x\| - \|y\| \ | \le \|x - y\| \le M \max \delta_i \le M \frac{\epsilon}{M} = \epsilon.$$

This means that we can make $\|y\|$ arbitrarily close to $\|x\|$ by making y close enough to x in the sense of the defined metric. Therefore $\|.\|$ is continuous. □

✍7 In $\mathbb{R}^n$ (or $\mathbb{C}^n$) all norms are equivalent.

Solution: We will do it for $\phi_1 = \|.\|$ some norm and $\phi_2 = \|.\|_\infty$ [and one can see that all other cases will follow from this one].

1. Need to show that for some α we have $\|x\| \le \alpha\|x\|_\infty$. Express x in the canonical basis of $\mathbb{R}^n$ as $x = \sum x_i e_i$ [look up canonical basis e_i from your csci2033 class.] Then

$$\|x\| = \|\sum x_i e_i\| \le \sum |x_i|\|e_i\| \le \max \|x_i\| \sum \|e_i\| \le \|x\|_\infty \alpha$$

where $\alpha = \sum \|e_i\|$.

2. We need to show that there is a β such that $\|x\| \ge \beta\|x\|_\infty$. Assume $x \ne 0$ and consider $u = x/\|x\|_\infty$. Note that u has infinity norm equal to one. Therefore it belongs to the closed and bounded set $S_\infty = \{v | \|v\|\infty = 1\}$. Since norms are continuous, the minimum of the norm $\|u\|$ for all $u's$ in $S_\infty$is *reached*, i.e., there is a $u_0 \in S_\infty$ such that

$$\min_{u \in S_\infty} \|u\| = \|u_0\|.$$

Let us call $\boldsymbol{\beta}$ this minimum value, i.e., $\|\boldsymbol{u}_0\| = \boldsymbol{\beta}$. Note in passing that $\boldsymbol{\beta}$ cannot be equal to zero otherwise $\boldsymbol{u}_0 = \boldsymbol{0}$ which would contradict the fact that $\boldsymbol{u}_0$ belongs to $\boldsymbol{S}_\infty$ [all vectors in $\boldsymbol{S}_\infty$ have infinity norm equal to one.] The result follows because $\boldsymbol{u} = \boldsymbol{x}/\|\boldsymbol{x}\|_\infty$, and so, remembering that $\boldsymbol{u} = \boldsymbol{x}/\|\boldsymbol{x}\|_\infty$, we obtain

$$\left\|\frac{\boldsymbol{x}}{\|\boldsymbol{x}\|_\infty}\right\| \geq \boldsymbol{\beta} \rightarrow \|\boldsymbol{x}\| \geq \boldsymbol{\beta}\|\boldsymbol{x}\|_\infty$$

This completes the proof □

✍14 Show that $\boldsymbol{\rho(A)} \leq \|\boldsymbol{A}\|$ for any matrix norm.

Solution: Let $\boldsymbol{\lambda}$ be the largest (in modulus) eigenvalue of $\boldsymbol{A}$ with associated eigenvector $\boldsymbol{u}$. Then

$$\boldsymbol{Au} = \boldsymbol{\lambda u} \rightarrow \frac{\|\boldsymbol{Au}\|}{\|\boldsymbol{u}\|} = |\boldsymbol{\lambda}| = \boldsymbol{\rho(A)}$$

This implies that

$$\boldsymbol{\rho(A)} \leq \max_{x \neq 0} \frac{\|\boldsymbol{Ax}\|}{\|\boldsymbol{x}\|} = \|\boldsymbol{A}\|$$

□

✍16 The eigenvalues of $\boldsymbol{A^H A}$ and $\boldsymbol{A A^H}$ are real nonnegative.

Solution: Let us show it for $\boldsymbol{A^H A}$ [the other case is similar] If $\boldsymbol{\lambda, u}$ is an eigenpair of $\boldsymbol{A^H A}$ then $\boldsymbol{(A^H A)u = \lambda u}$. Take inner products with $\boldsymbol{u}$ on both sides. Then:

$$\boldsymbol{\lambda(u, u) = ((A^H A)u, u) = (Au, Au) = \|Au\|^2}$$

Therefore, $\boldsymbol{\lambda = \|Au\|^2 / \|u\|^2}$ which is a real nonnegative number. $\square$

[Note: 1) Observe how simple the proof is for such an important fact. It is based on the result $\boldsymbol{(Ax, y) = (x, A^H y)}$. 2) The singular values of $\boldsymbol{A}$ are the square roots of the eigenvalues of $\boldsymbol{A^H A}$ if $\boldsymbol{m \geq n}$ or those of the eigenvalues of $\boldsymbol{A A^H}$ if $\boldsymbol{m < n}$. So there are always $\boldsymbol{\min(m, n)}$ singular values. This is really just a preliminary definition as we need to refer to singular values often – but we will see singular values and the singular value decomposition in great detail later.]

✍17 Prove that when $\boldsymbol{A} = \boldsymbol{u}\boldsymbol{v}^T$ then $\|\boldsymbol{A}\|_2 = \|\boldsymbol{u}\|_2\|\boldsymbol{v}\|_2$.

Solution: Done in class. We start by dealing the eigenvalues of an arbitrary matrix of the form $\boldsymbol{A} = \boldsymbol{u}\boldsymbol{v}^T$ where both $\boldsymbol{u}$ and $\boldsymbol{v}$ are in $\mathbb{R}^n$. From $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{\lambda}\boldsymbol{x}$ we get:

$$\boldsymbol{u}\boldsymbol{v}^T\boldsymbol{x} = \boldsymbol{\lambda}\boldsymbol{x} \rightarrow (\boldsymbol{v}^T\boldsymbol{x})\boldsymbol{u} = \boldsymbol{\lambda}\boldsymbol{x}$$

Notice that we did this because $\boldsymbol{v}^T\boldsymbol{x}$ is a scalar. We have 2 cases.

Case 1: $\boldsymbol{v}^T\boldsymbol{x} = \boldsymbol{0}$. In this case it is clear that the equation $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{\lambda}\boldsymbol{x}$ is satisfied with $\boldsymbol{\lambda} = \boldsymbol{0}$. So any vector that is orthogonal to $\boldsymbol{v}$ is an eigenvector of $\boldsymbol{A}$ associated with the eigenvalue $\boldsymbol{\lambda} = \boldsymbol{0}$. (It can be shown that the eigenvalue 0 is of multiplicity $\boldsymbol{n} - \boldsymbol{1}$).

Case 2: $\boldsymbol{v}^T\boldsymbol{x} \neq \boldsymbol{0}$. In this case it is clear that the equation $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{\lambda}\boldsymbol{x}$ is satisfied with $\boldsymbol{\lambda} = \boldsymbol{v}^T\boldsymbol{u}$ and $\boldsymbol{x} = \boldsymbol{u}$. So $\boldsymbol{u}$ is an eigenvector of $\boldsymbol{A}$ associated with the eigenvalue $\boldsymbol{v}^T\boldsymbol{x}$.

In summary the matrix $\boldsymbol{u}\boldsymbol{v}^T$ has only two eigenvalues: 0, and $\boldsymbol{v}^T\boldsymbol{u}$.

Going back to the original question, we consider now $\boldsymbol{A} = \boldsymbol{u}\boldsymbol{v}^T$ and we are interested in the

2-norm of $\boldsymbol{A}$. We have

$$\|\boldsymbol{A}\|_2^2 = \rho(\boldsymbol{A}^T\boldsymbol{A}) = \rho(\boldsymbol{v}\boldsymbol{u}^T\boldsymbol{u}\boldsymbol{v}^T) = \|\boldsymbol{u}\|_2^2\rho(\boldsymbol{v}\boldsymbol{v}^T) = \|\boldsymbol{u}\|_2^2\|\boldsymbol{v}\|_2^2.$$

The last relation comes from what was done above to determine the eigenvalues of $\boldsymbol{v}\boldsymbol{v}^T$. $\square$