✍1 Show that $\kappa(I) = 1$;

Solution: This is obvious because for any matrix norm $\|A\| = \|A^{-1}\| = 1$. $\square$

✍2 Show that $\kappa(A) \geq 1$;

Solution: We have $\|AA^{-1}\| = \|I\| = 1$ therefore $1 = \|AA^{-1}\| \leq \|A\| \; \|A^{-1}\| = \kappa(A)$ $\square$

✍5 Show that if $\|E\| \leq \delta$ and $\|e_b\|/\|b\| \leq \delta$ then

$$\frac{\|x-y\|}{\|x\|} \leq \frac{2\delta\kappa(A)}{1-\delta\kappa(A)}$$

Solution: From the main theorem (theorem 1) we have

$$\frac{\|x-y\|}{\|x\|} \leq \frac{\|A^{-1}\| \; \|A\|}{1-\|A^{-1}\| \; \|E\|} \left(\frac{\|E\|}{\|A\|} + \frac{\|e_b\|}{\|b\|}\right)$$

If $\|\boldsymbol{E}\| \le \boldsymbol{\delta}$ and $\|\boldsymbol{e_b}\|/\|\boldsymbol{b}\| \le \boldsymbol{\delta}$ then:

$$\begin{aligned}\frac{\|\boldsymbol{x}-\boldsymbol{y}\|}{\|\boldsymbol{x}\|} &\le \frac{\boldsymbol{\kappa}(\boldsymbol{A}) \times 2\boldsymbol{\delta}}{1 - \|\boldsymbol{A}^{-1}\| \ \|\boldsymbol{E}\|} \\ &\le \frac{2\boldsymbol{\delta}\boldsymbol{\kappa}(\boldsymbol{A})}{1 - \|\boldsymbol{A}^{-1}\|\|\boldsymbol{A}\| \times (\|\boldsymbol{E}\|/\|\boldsymbol{A}\|)} \\ &\le \frac{2\boldsymbol{\delta}\boldsymbol{\kappa}(\boldsymbol{A})}{1 - \boldsymbol{\delta}\boldsymbol{\kappa}(\boldsymbol{A})}.\end{aligned}$$

□

✍9 Show that $\frac{\|\boldsymbol{x}-\tilde{\boldsymbol{x}}\|}{\|\boldsymbol{x}\|} \ge \frac{1}{\boldsymbol{\kappa}(\boldsymbol{A})} \frac{\|\boldsymbol{r}\|}{\|\boldsymbol{b}\|}$.

Solution: As before we start with noting that $\boldsymbol{A}(\boldsymbol{x}-\tilde{\boldsymbol{x}}) = \boldsymbol{b} - \boldsymbol{A}\tilde{\boldsymbol{x}} = \boldsymbol{r}$. So:

$$\|\boldsymbol{r}\| \le \|\boldsymbol{A}\|\|\boldsymbol{x}-\tilde{\boldsymbol{x}}\| \to \frac{\|\boldsymbol{r}\|}{\|\boldsymbol{b}\|} \le \|\boldsymbol{A}\|\frac{\|\boldsymbol{x}-\tilde{\boldsymbol{x}}\|}{\|\boldsymbol{b}\|}$$

Next from $\|\boldsymbol{x}\| = \|\boldsymbol{A}^{-1}\boldsymbol{b}\| \le \|\boldsymbol{A}^{-1}\|\|\boldsymbol{b}\|$ we get $\|\boldsymbol{b}\| \ge \|\boldsymbol{x}\|/\|\boldsymbol{A}^{-1}\|$ and so

$$\frac{\|\boldsymbol{r}\|}{\|\boldsymbol{b}\|} \le \|\boldsymbol{A}\|\frac{\|\boldsymbol{x}-\tilde{\boldsymbol{x}}\|}{\|\boldsymbol{x}\|/\|\boldsymbol{A}^{-1}\|} = \boldsymbol{\kappa}(\boldsymbol{A})\frac{\|\boldsymbol{x}-\tilde{\boldsymbol{x}}\|}{\|\boldsymbol{x}\|}$$

which yields the result after dividing the 2 sides by $\boldsymbol{\kappa}(\boldsymbol{A})$. □

Proof of Theorem 3

Let $\boldsymbol{D} \equiv \|\boldsymbol{E}\|\|\boldsymbol{y}\| + \|\boldsymbol{e_b}\|$ and $\boldsymbol{\eta} \equiv \boldsymbol{\eta_{E,e_b}(y)}$. The theorem states that $\boldsymbol{\eta} = \|\boldsymbol{r}\|/\boldsymbol{D}$. Proof in 2 steps.

First: Any $\boldsymbol{\Delta A}, \boldsymbol{\Delta b}$ pair satisfying (1) is such that $\boldsymbol{\epsilon} \geq \|\boldsymbol{r}\|/\boldsymbol{D}$. Indeed from (1) we have (recall that $\boldsymbol{r} = \boldsymbol{b} - \boldsymbol{Ay}$)

$$\boldsymbol{Ay} + \boldsymbol{\Delta Ay} = \boldsymbol{b} + \boldsymbol{\Delta b} \rightarrow \boldsymbol{r} = \boldsymbol{\Delta Ay} - \boldsymbol{\Delta b} \rightarrow$$

$$\|\boldsymbol{r}\| \leq \|\boldsymbol{\Delta A}\|\|\boldsymbol{y}\| + \|\boldsymbol{\Delta b}\| \leq \boldsymbol{\epsilon}(\|\boldsymbol{E}\|\|\boldsymbol{y}\| + \|\boldsymbol{e_b}\|) \rightarrow \boldsymbol{\epsilon} \geq \frac{\|\boldsymbol{r}\|}{\boldsymbol{D}}$$

Second: We need to show an instance where the minimum value of $\|\boldsymbol{r}\|/\boldsymbol{D}$ is reached. Take the pair $\boldsymbol{\Delta A}, \boldsymbol{\Delta b}$:

$$\boldsymbol{\Delta A} = \boldsymbol{\alpha r z^T}; \quad \boldsymbol{\Delta b} = \boldsymbol{\beta r} \quad \text{with } \boldsymbol{\alpha} = \frac{\|\boldsymbol{E}\|\|\boldsymbol{y}\|}{\boldsymbol{D}}; \quad \boldsymbol{\beta} = \frac{\|\boldsymbol{e_b}\|}{\boldsymbol{D}}$$

The vector z depends on the norm used - for the 2-norm: $z = y/\|y\|^2$. Here: Proof only for 2-norm

a) We need to verify that first part of (1) is satisfied:

$$\begin{aligned}(A + \Delta A)y &= Ay + \alpha r \frac{y^T}{\|y\|^2} y = b - r + \alpha r \\ &= b - (1-\alpha) r = b - \left(1 - \frac{\|E\| \, \|y\|}{\|E\| \, \|y\| + \|e_b\|}\right) r \\ &= b - \frac{\|e_b\|}{D} r = b + \beta r \quad \rightarrow \\ (A + \Delta A)y &= b + \Delta b \quad \leftarrow \text{The desired result}\end{aligned}$$

Finally: b) Must now verify that $\|\Delta A\| = \eta\|E\|$ and $\|\Delta b\| = \eta\|e_b\|$. Exercise: Show that $\|uv^T\|_2 = \|u\|_2\|v\|_2$

$$\|\Delta A\| = \frac{|\alpha|}{\|y\|^2}\|ry^T\| = \frac{\|E\|\,\|y\|}{D}\frac{\|r\|\,\|y\|}{\|y\|^2} = \eta\|E\|$$

$$\|\Delta b\| = |\beta|\|r\| = \frac{\|e_b\|}{D}\|r\| = \eta\|e_b\| \qquad QED$$