✍1 Show that each $\boldsymbol{A_k}$ [$\boldsymbol{A(1:k,1:k)}$ in matlab notation] is SPD.

Solution: Let $\boldsymbol{x}$ be any vector in $\mathbb{R}^k$ and consider the vector $\boldsymbol{y}$ of $\mathbb{R}^n$ obtained by stacking $\boldsymbol{x}$ followed by $\boldsymbol{n-k}$ zeros. Then it can be easily seen that : $\boldsymbol{(A_k x, x) = (Ay, y)}$ and since $\boldsymbol{A}$ is SPD then $\boldsymbol{(Ay, y) > 0}$ and therefore $\boldsymbol{(A_k x, x) > 0}$ for any $\boldsymbol{x}$ in $\mathbb{R}^k$. Hence $\boldsymbol{A_k}$ is SPD. □

✍2 Consequence $\boldsymbol{\det(A_k) > 0}$

Solution: This is because the determinant is the product of the eigenvalues which are real positive (see notes). □

✍3 If $\boldsymbol{A}$ is SPD then for any $\boldsymbol{n \times k}$ matrix $\boldsymbol{X}$ of rank $\boldsymbol{k}$, the matrix $\boldsymbol{X^T A X}$ is SPD.

Solution: For any $\boldsymbol{v} \in \mathbb{R}^k$ we have $\boldsymbol{(X^T A X v, v) = (AXv, Xv)}$. In addition, since $\boldsymbol{X}$ is of full rank, then $\boldsymbol{Xv}$ cannot be zero if $\boldsymbol{v}$ is nonzero. Therefore we have $\boldsymbol{(AXv, Xv) > 0}$. □

✍ 4 Show that if $\boldsymbol{A}^T = \boldsymbol{A}$ and $(\boldsymbol{Ax}, \boldsymbol{x}) = \boldsymbol{0}\ \forall \boldsymbol{x}$ then $\boldsymbol{A} = \boldsymbol{0}$.

Solution: The condition implies that for all $\boldsymbol{x}, \boldsymbol{y}$ we have $(\boldsymbol{A}(\boldsymbol{x}+\boldsymbol{y}), \boldsymbol{x}+\boldsymbol{y}) = \boldsymbol{0}$. Now expand this as: $(\boldsymbol{Ax}, \boldsymbol{x})+(\boldsymbol{Ay}, \boldsymbol{y})+2(\boldsymbol{Ax}, \boldsymbol{y}) = \boldsymbol{0}$ for all $\boldsymbol{x}, \boldsymbol{y}$ which shows that $(\boldsymbol{Ax}, \boldsymbol{y}) = 0\ \forall\ \boldsymbol{x}, \boldsymbol{y}$. This implies that $\boldsymbol{A} = \boldsymbol{0}$ (e.g. take $\boldsymbol{x} = \boldsymbol{e}_j, \boldsymbol{y} = \boldsymbol{e}_i$)... ☐

✍ 5 Show: A nonzero matrix $\boldsymbol{A}$ is indefinite iff $\exists\ \boldsymbol{x}, \boldsymbol{y} : (\boldsymbol{Ax}, \boldsymbol{x})(\boldsymbol{Ay}, \boldsymbol{y}) < \boldsymbol{0}$.

Solution:

[←] Trivial. The matrix cant be PSD or NSD under the conditon

[→] Need to prove: If $\boldsymbol{A}$ is indefinite then there exist such that $\boldsymbol{x}, \boldsymbol{y} : (\boldsymbol{Ax}, \boldsymbol{x})(\boldsymbol{Ay}, \boldsymbol{y}) < \boldsymbol{0}$. Assume contrary is true, i.e., $\forall \boldsymbol{x}, \boldsymbol{y} (\boldsymbol{Ax}, \boldsymbol{x})(\boldsymbol{Ay}, \boldsymbol{y}) \geq \boldsymbol{0}$. There is at least one $\boldsymbol{x}_0$ such that $(\boldsymbol{Ax}_0, \boldsymbol{x}_0)$ is nonzero, otherwise $\boldsymbol{A} = \boldsymbol{0}$ from previous question. Assume $(\boldsymbol{Ax}_0, \boldsymbol{x}_0) > \boldsymbol{0}$. Then $\forall \boldsymbol{y} (\boldsymbol{Ax}_0, \boldsymbol{x}_0)(\boldsymbol{Ay}, \boldsymbol{y}) \geq \boldsymbol{0}$. which implies $\forall \boldsymbol{y} : (\boldsymbol{Ay}, \boldsymbol{y}) \geq \boldsymbol{0}$, i.e., $\boldsymbol{A}$ is positive semi-definite. This contradicts the assumption that $\boldsymbol{A}$ is neither positive nor negative semi-defininte ☐