✍1 How would you get an orthonormal basis of $\boldsymbol{X}$?

Solution: Just take any basis and orthonormalize it with Gram-Schmidt of Householder QR.

□

✍2 Show how you can get a decomposition in which $\boldsymbol{C}$ is lower (or upper) triangular, from the above factorization.

Solution: You first get any factorization in the form shown in Page 9-7 – Then to get an upper triangular $\boldsymbol{C}$ you use the QR factorization $\boldsymbol{C} = \boldsymbol{QR}$. Then $\boldsymbol{U}$ is replaced by

$$\boldsymbol{U_{new}} = \boldsymbol{U} \times \begin{pmatrix} \boldsymbol{Q} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{I} \end{pmatrix}$$

and $\boldsymbol{C}$ is replaced by $\boldsymbol{R}$. To get a lower triangular $\boldsymbol{C}$ you can use the same trick applied to $\boldsymbol{A^T}$ and transpose the final result. □

✍3 How can you get the ULV decomposition by using only the Householder QR factorization (possibly with pivoting)?

Solution: You first get the Householder QR factorization $\boldsymbol{A} = \boldsymbol{Q_1 R_1}$ of the matrix $\boldsymbol{A}$. The second step is to perform a Householder QR factorization of the matrix $\boldsymbol{R_1^T}$, so you will get: $\boldsymbol{R_1^T} = \boldsymbol{Q_2 R_2}$. The final step is to write:

$$\boldsymbol{A} = \boldsymbol{Q_1} * \boldsymbol{R_2^T} * \boldsymbol{Q_2^T} \equiv \boldsymbol{URV^T}$$

where $\boldsymbol{U} = \boldsymbol{Q_1} \in \mathbb{R}^{m \times m}$; $\boldsymbol{V} = \boldsymbol{Q_2} \in \mathbb{R}^{n \times n}$; $\boldsymbol{R} = \boldsymbol{R_2} \in \mathbb{R}^{m \times n}$ □

✍4 In the proof of the SVD decomposition, define $\boldsymbol{U, V}$ as single Householder reflectors.

Solution: We deal with $\boldsymbol{U}$ only [proceed similarly with $\boldsymbol{V}$]. We need a matrix $\boldsymbol{P} = \boldsymbol{I} - 2\boldsymbol{ww^T}$ such that the first column of $\boldsymbol{A}$ is $\boldsymbol{u_1}$ and all columns are orthonormal. The second requirement is satisfied by default since $\boldsymbol{P}$ is unitary. Note that what if $\boldsymbol{P}$ is available we will have $\boldsymbol{Pu_1} = \boldsymbol{e_1}$ because $\boldsymbol{P^2} = \boldsymbol{I}$. Therefore, the wanted $\boldsymbol{w}$ is simply the vector that transforms the vector $\boldsymbol{u_1}$ into $\boldsymbol{\alpha e_1}$. ... □

✍ 5 How can you obtain the thin SVD from the QR factorization of $\boldsymbol{A}$ and the SVD of an $\boldsymbol{n} \times \boldsymbol{n}$ matrix?

Solution: We first get the thin QR factorization of $\boldsymbol{A}$, namely $\boldsymbol{A} = \boldsymbol{Q}\boldsymbol{R}$ where $\boldsymbol{Q} \in \mathbb{R}^{\boldsymbol{m}\times\boldsymbol{n}}$ and $\boldsymbol{R} \in \mathbb{R}^{\boldsymbol{n}\times\boldsymbol{n}}$; Then we can get the SVD $\boldsymbol{R} = \boldsymbol{U_R}\boldsymbol{\Sigma}\boldsymbol{V_R^T}$ of $\boldsymbol{R}$ and this yields:

$$\boldsymbol{A} = \boldsymbol{Q} \times \boldsymbol{U_R}\boldsymbol{\Sigma_R}\boldsymbol{V_R^T} \rightarrow \boldsymbol{A} = \boldsymbol{U}\boldsymbol{\Sigma}\boldsymbol{V^T}, \quad \text{with} \quad \boldsymbol{U} = \boldsymbol{Q} \times \boldsymbol{U_R}; \quad \boldsymbol{\Sigma} = \boldsymbol{\Sigma_R}; \quad \boldsymbol{V} = \boldsymbol{V_R}.$$