$\| A \| = \max \|Ax\| / \| x \|$ ==> $\| A x \| \leq \| A \| \|x\|$

$\|A B \| = \max_{\|x\|=1} \|A B x\| \leq \max_{\|x\|=1} \|A \| \| B x\|$

$\leq \|A \| \max_{\|x\|=1} \| B x\|$

$\leq \|A \| \| B \|$

In particular: For an nxn matrix A

$\|A^k\| \leq \|A\|^k$

$\|A \|<1$ ===> $\|A^k\| \to 0$

===

proof of expression of $\| A \|_1$

$\| A \|_1 = \max_{\|x\|_1 =1} \| Ax \|_1$

Define $\eta = \max_j \| A(:,j)\|_1 = \|A(:,j0)\|_1$

1) $\|A\|_1 \leq \eta$
let x s.t. $\|x\|_1=1$

$\| Ax\|_1 = \| \sum x_j A(:,j) \|_1 \leq \sum_j |x_j| \| A(:,j)\|_1$

$\leq \sum_j |x_j| \max_j \| A(:,j)\|_1 = \sum |x_j| \eta$

$\| Ax\|_1 \leq \eta \sum_j |x_j| = \eta$

==> $\|A\|_1 \leq \eta$

2) $\|A\|_1 \geq \eta$

Assume max reached for j0
let x = e_{j0} = [0; 0; ..., 0; 1; 0 ....]
^- position j0

$A e_j$ = jth column of A
j = j0 ==> $|| A (:,j0)||_1 = \max || A(:,j) ||_1$

Then $\| A x \|_1 = \| A(:,j0) \|_1 = \eta$

For one particular x $\| A x\|_1 \geq \eta$

$\max \| A x\|_1 \geq \eta$

==> $\|A\|_1 \geq \eta$

===

$\rho(A) \leq \|A\|$??

Let λ be the largest eigenvalue in modulus and write
with $\|u\| = 1$
‖

$A\,u = \lambda\,u$
$\|A\,u\| = |\lambda|\ \|u\| = |\lambda| = \rho(A)$

$\|A\,u\| \leq \|A\|\ \|u\| = \|A\|$ ==> $\|A\| \geq \rho(A)$