Householder transformation:

Properties: [Note: $\| . \| = \| . \|_2$]

$P = I - 2 w w^T$ with $\| w \| = 1$

$P^T = P$ ==> P is symmetric

$P^T P = (I - 2 w w^T)(I - 2 w w^T) = I - 2 w w^T - 2 w w^T + 4 w \overbrace{w^T w}^{= 1} w^T$

$I - 2 w w^T - 2 w w^T + 4 w \; w^T = I$

==> P = unitary

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Alternative notation:

write P as

$P = I - \beta v v^T$ where v need not be a unit vector
and

$\beta = 2/(v^T v) = 2 / \| v\|^2$.

Question: how to use this in practice?

How to compute P x ?

$(I - \beta v v^T) x = x - \beta v v^T x = x - (\beta v^T x) \; v$

steps:
1) compute: $s = \beta \underbrace{(v^T x)}_{2m}$ cost: 2m +1
2) compute: $x - s v$: 2m

total cost: 4m+1 ~ 4m

What about P *A?

$(I - \beta v v^T) A = A - v z^T$

where $z^T = \beta v^T A$

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Question: find v such that

$(I - \beta v v^T) x = \alpha e_1$

$x - \beta v v^T x = x - \underbrace{(\beta v^T x)}_{s} v = \alpha e_1$ ==>

$x - s \; v = \alpha e_1$ ==> $s v = x - \alpha e_1$

$\alpha = ?$

From $\|(I - \beta v v^T) x\| = \|\alpha e_1\| = | \alpha |$ ==> $| \alpha | = \| x \|$

$\alpha = \pm \| x \|$ ==> both signs work

answer to the question

$v = x \pm \| x \| e_1$ $e_1 = [1, 0,0,....,0]^T$

$\beta = 2 / || v ||^2$

Let $v = [\eta_1\ \eta_2\ ....\ \eta_m]^T$
$z = [\xi_1\ \xi_2\ ....\ \xi_m]$

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which sign is better? $v = x - \alpha e_1$
If I choose the + sign

$\eta_1 = \xi_1 + \| x \|$ $\eta_i = \xi_i$ $i>1$ better if $\xi_1 > 0$

Otherwise

$\eta_1 = \xi_1 - \| x \|$ $\eta_i = \xi_i$ $i>1$ $\xi_1 < 0$

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Alternative//

Use this in all cases:

$\eta_1 = \xi_1 - \| x \|$

= when $\xi_1 <0$ == no problem
= otherwise compute $\eta_1 = \xi_1 - \| x \|$ as

$\eta_1 = (\xi_1 - \| x \|)(\xi_1 + \| x \|) /(\xi_1 + \| x \|)$

$= (\xi_1^2 - \| x \|^2) /(\xi_1 + \| x \|)$

$= [\xi_1^2 - [\xi_1^2 + \xi_2^2 .... + \xi_m^2]] /(\xi_1 + \| x \|)$

$= - [\xi_2^2 + \xi_3^2 .... + \xi_m^2] /(\xi_1 + \| x \|)$

$= - \sigma / (\xi_1 + \| x \|)$

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Matlab demo [marathon regression]

X = [x1 x2 x3]
b = times column
a = $[\alpha_1\ \alpha_2\ \alpha_3]^T$

$X a \approx b$ in least-squares sense : $\min \| b - X a \|_2$

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