Householder

[Note: $\| . \| = \| . \|_2$]

$P = I - 2 w w^T$ with $\| w \| = 1$

or

$P = I - \beta v v^T$ with $\beta = 2 / v^T v$

==

Question 1: given x find v s.t. $Px = \alpha e_1$

$(I-\beta v v^T)x = \alpha e_1$

Answer: $v = x - \alpha e_1$ with $\alpha = \pm \| x \|$ [both signs work]

---> two different implementations

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NEXT: generalization

Question 2: Given

$x = \begin{vmatrix} x_1 \\ x_2 \end{vmatrix}$ $x_1 \in R^k$ $x_2 \in \mathbb{R}^{m-k}$

find v such that $P v = \begin{vmatrix} x_1 \\ y \end{vmatrix}$ with $y = \alpha e_1 \in \mathbb{R}^{m-k}$

solution: select v as follows:

$v = \begin{vmatrix} v_1 \\ v_2 \end{vmatrix}$ set $v_1 = 0 \Rightarrow v = \begin{vmatrix} 0 \\ v_2 \end{vmatrix}$

Px =?

$x - \underbrace{\beta (v^T x)}_{\text{scalar } s} v = \begin{vmatrix} x_1 \\ y \end{vmatrix}$

==================================

$y = x_2 - s * v_2$ $s = \beta v_2^T x_2$

==> everything as if we work only on second part (x_2)

Obtain v_2 as a Householder vector to transform x_2 into αe_1

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$X_1 = P_1 X$

$X_2 = P_2 X_1 = P_1 P_1 X$

$X_3 = P_3 X_2 = ...$

⋮

$X_n = P_n X_{n-1} = P_n P_{n-1} .... P_1 X$ = upper triangular ≡ R

=============== Apply inverse of $P_n P_{n-1} \dots P_1$ on left:

$[P_n P_{n-1} \dots P_1]^{-1} = P_1^{-1} \times P_2^{-1} \dots P_n^{-1} = P_1 P_2 \dots P_n \equiv Q$

$[P_i^{-1} = P_i]$
==>

$X = Q R$

X is $m \times n$

Differences with Gram-Schmidt:
* here Q is of size : $m \times m$
* R is of size : $m \times n$ – R is upper triangular.

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How to solve LS problems?

Important : you never form Q explicitly! [$m \times m$ matrix – expensive]

Want to min $\| \overbrace{Q R}^{A} x - b \|$ == $\min \| Q^T (Q R x - b) \| = \min \| R x - Q^T b \|$

$R = \begin{vmatrix} R_1 \\ 0 \end{vmatrix} \quad Q^T b = c = \begin{vmatrix} c_1 \\ c_2 \end{vmatrix}$

$\left\| \begin{vmatrix} R_1 \\ 0 \end{vmatrix} x - \begin{vmatrix} c_1 \\ c_2 \end{vmatrix} \right\|^2 = \| R_1 x - c_1 \|^2 + \| c_2 \|^2$

solve $R_1 x = c_1$ ==> Done