Background on orth. decomposition and URV

1)

X a subspace of $\mathbb{R}^m$ then [orthogonal decomposition]:

$\mathbb{R}^m = X \oplus X\perp$

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2) $A \in \mathbb{R}^{m \times n}$ m rows n columns [often m>n – but in this case m<n]

e.g. m = 3 , n = 5 :

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    x x x x x
A=  x x x x x
    x x x x x
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Let X = Ran(A) Then: $\mathbb{R}^m = X \oplus X\perp = \text{Ran}(A) \oplus \text{Ran}(A)\perp$

Observe that: $\text{Ran}(A)\perp = \text{Null}(A^T)$

so:

$\mathbb{R}^m = X \oplus X\perp$ X is a subspace of $\mathbb{R}^m$

$\mathbb{R}^m = \text{Ran}(A) \oplus \text{Null}(A^T)$
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Do the same thing for A^T:

$\mathbb{R}^n = \text{Ran}(A^T) \oplus \text{Null}(A)$
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3) Express A in bases for $\mathbb{R}^m$ and $\mathbb{R}^n$ ... ==> URV

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Pr. Ex. # 08 :

Q: What are all solutions of system

Ax = b

when m<n .. [assume A has rank m]

Find the solution x_s of smallest length.

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$x \in \mathbb{R}^n = x_1 + x_2$
where $x_1 \in \text{Ran}(A^T)$ and $x_2 \in \text{Null}(A)$

$b \in \mathbb{R}^m = b_1 + b_2$
A^T is n x m n>m has rank n [full column rank]
$\mathbb{R}^m = \text{Ran}(A) \oplus \underline{\text{Null}(A^T)} = \{0\}$

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how to get x_1? $x_1 \in$ Ran(A^T)   ==> x_1 in the span of columns of A^T
we can write: $x_1 = A^T y$ where $y \in \mathbb{R}^m$
$A x = b$ ==>

$A [x_1 + x_2] = b$ ==> $A x_1 = b$ ==> $A A^T y = b$ ==> solve $(AA^T) y = b$
—– x ——   ——— size =m x m invertible
$x_1 \in$ Ran(A^T) and $x_2 \in$ Null(A)

[recall that when A is mxn m>n of full rank then $A^T A$ is invertible]

what are all solutions?

$x = x_1 + x_2$ where $x_1 = A^T y$ [y unique] and x_2 *any* vector of null(A)

what is dimension of null(A)?

n-m

q: which of these solutions has the smallest norm?

$x = x_1 + x_2$

$\| x \|^2 = \| x_1\|^2 + \| x_2\|^2$

norm is min when $x_2 = 0$

$x_s = x_1$ is solution with smallest norm.

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How to compute x_s ?

solution 1 : use the above.

solve $(A A^T) y = b$
then $x_s = A^T y$

Solution 2 :

$A^T = Q R$ [e.g. Gram-Schmidt]
Then write solution as $x_1 = Q y$

$x_1 = Q y$

$A (Q y) = b$ ==> $(R^T Q^T) Q y = b$ ==> $R^T y = b$ ==>
solve for y....

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Proof of SVD decomposition
all norms $\| \|$ are 2-norms ..

$A \in \mathbb{R}^{m \times n}$

1)
Let $\sigma_1 = \| A \| =$ max of $\| A x\|$ for all vectors x $\| x\|=1$

$\sigma_1 = \| A v_1\|$ with $\| v_1 \|=1$

Let $u_1 = A v_1 / \sigma_1$ Note : $\| u_1 \| = 1$

2) Complete u_1 into a basis of $\mathbb{R}^m$:

$U = [u_1, u_2 \; .... \; u_m]$ unitary

Complete u_1 into a basis of $\mathbb{R}^n$
$V = [v_1, v_2 \; .... \; v_n]$ Unitary

3)

Consider $U^T A V = \begin{vmatrix} \sigma_1 & w^T \\ 0 & B \end{vmatrix}$ call this matrix A_1

4) Claim w must be equal to zero.

Let $x = \begin{vmatrix} \sigma_1 \\ w \end{vmatrix}$

compute $A_1 x = \begin{vmatrix} \sigma_1^2 + w^T w \\ B w \end{vmatrix}$

Contradiction argument: assume $w \neq 0$ then

$\| x \| \geq |x_1|$ ==>

$\| A_1 x \| \geq [\sigma_1^2 + w^T w] = \sqrt{[\sigma_1^2 + w^T w]} \; \| x \| > \sigma_1 \| x \|$ (*)

[recall: $\| x \| = \sqrt{[\sigma_1^2 + w^T w]}$]

why is (*) a contradiction?
answer : $\| A_1 \| = \| A \| = \sigma_1$

5) Consider $U^T A V = \begin{vmatrix} \sigma_1 & 0 \\ 0 & B \end{vmatrix}$

Induction argument: $B = U_1 \Sigma_1 V_1^T$

then $A = U \begin{vmatrix} \sigma_1 & 0 \\ 0 & U_1 \Sigma_1 V_1^T \end{vmatrix} V^T$

$U\sim = \begin{vmatrix} 1 & 0 \\ 0 & U_1 \end{vmatrix}$

$A = U \, U\sim \; \Sigma \, (V \, V\sim)^T$