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The SVD

$A \in \mathbb{R}^{m \times n}$

$A = U \Sigma V^T \quad = U \in \mathbb{R}^{m \times m}, V \in \mathbb{R}^{n \times n}, \Sigma \in \mathbb{R}^{m \times n}$

U, V, unitary, Σ diagonal

$$\Sigma = ?\begin{vmatrix} \sigma_1 & & & \\ & \sigma_2 & & ? \\ & & \ddots & ? \\ & & & \sigma_p ? \\ & & & 0 \\ & & & \vdots \end{vmatrix} \qquad p = \min(m,n)$$

$[U_1\ U_2]\ \Sigma\ V^T == U_1\ \Sigma_1\ V^T \quad$ ==> thin SVD

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alternative expression of SVD

$$A = U \Sigma V^T = \overset{\overset{r}{\longleftrightarrow}}{[U_1\ U_2]} \begin{vmatrix} \Sigma_1 & 0 \\ 0 & 0 \end{vmatrix} \begin{vmatrix} V_1^T \\ V_2^T \end{vmatrix} = U_1\ \Sigma_1\ V_1^T$$

? ?

$$\begin{matrix} & \begin{vmatrix} \sigma_1\ v_1^T \\ \sigma_2\ v_2^T \\ \vdots \\ \sigma_r\ v_r^T ? \end{vmatrix} \\ \begin{matrix} [u_1 & u_2 & \cdots & u_r] \\ \vdots & & & \vdots \end{matrix} & \end{matrix}$$

$= \sigma_1\ u_1\ v_1^T + \ldots + \sigma_r\ u_r\ v_r^T$

===== outer product (rank one) matrix

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SVD and a few expressions of norms

$\|A\| = \|\ U \Sigma V^T\ \| = \|\ \Sigma V^T\ \| = \|\ \Sigma\ \| = \sigma_1$

[$\|\ \Sigma x\ \|^2 = \sum \sigma_i^{\ 2} x_i^2 \le \sigma_1^2$ .... reached...]

$\|\ A\ \|_F = \|\ \Sigma\ \|_F$ ...

A is nxn invertible:

$\|A^{-1}\| = \|\ V \Sigma^{-1} U^T\ \| = \|\ \Sigma^{-1}\ \| = 1/\sigma_n$

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Proof of Eckhart Young theorem

$X = \text{span}\{v_1, \ldots, v_{k+1}\}$ => $\dim(X) = k+1$

$\text{null}(B)$ => $\dim(\text{Null}(B)) = n-k$

claim: $X \cap \text{Null}(B) \neq \{0\}$

Let $x_0 \in X \cap \text{Null}(B)$

write in basis V:

$x_0 = V y \qquad y = \begin{vmatrix} y_1 \\ 0 \end{vmatrix}$ r+1 components

assume $\| x_0 \| = 1$ ==> $\| y_1 \| = 1$

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1) $\| A - B \| \geq \sigma_{k+1}$?

$$\| A - B \| \geq \| (A - B)x_0 \| = \| A x_0 \| = \| U \Sigma V^T x_0 \| = \| \Sigma y \|$$

$$\| \Sigma y \|^2 = \sum_{i=1}^{k+1} \sigma_i^2 y_i^2 \geq \sum \sigma_{k+1}^2 y_i^2 = \sigma_{k+1}^2 \sum y_i^2 = (\sigma_{k+1})^2$$

$\| A - B \| \geq \sigma_{k+1}$

2) $\| A - B \| = \sigma_{k+1}$?

Take $B = A_k$

$$\| A - A_k \| = \left\| \sum_{i=k+1}^{r} \sigma_i u_i v_i^T \right\| = \sigma_{k+1}$$

Done.

□

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Variation of the result:

$\min_{\{B, \text{rank}(B) = k\}} \| A - B \| = \sigma_{k+1}$

also true:

$\min_{\{B, \text{rank}(B) \leq k\}} \| A - B \| = \sigma_{k+1}$

$\min_{\{B, \text{rank}(B) \leq j\}} \| A - B \| = \sigma_{j+1} \geq \sigma_{k+1}$

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$A v_i = \sigma_i u_i$??

$$[\ \sigma_1 u_1 v_1^T + \ldots + \sigma_r u_r v_r^T\ ]\ v_i = \sigma_i u_i v_i^T v_i = \sigma_i u_i$$

$A^T u_i = \sigma_i v_i$

$A^T(A v_i) = A^T (\sigma_i u_i) = \sigma_i^2 v_i$

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Example

$$A = \begin{vmatrix} 1 & 0 & 2 & 0 \\ 0 & 0 & -2 & 1 \end{vmatrix}$$

Singular values?

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compute: $A A^T =$

$$A A^T = ? \begin{vmatrix} 5 & -4 \\ -4 & 5 \end{vmatrix}$$

$\det(A A^T - \lambda I) = (5-\lambda)^2 - 16 \implies \lambda = 5 \pm 4$

$\sigma_1 = 3$

$\sigma_2 = 1$

$$\lambda=9 \Rightarrow \quad AA^T - \lambda I = ? \begin{vmatrix} -4 & -4 \\ -4 & -4 \end{vmatrix}$$

$$u_1 = ? \begin{vmatrix} 1 \\ -1 \end{vmatrix} / \sqrt{2}$$

$$u_2 = ? \begin{vmatrix} 1 \\ 1 \end{vmatrix} / \sqrt{2}$$

$U = [u_1 \; u_2]$

$A = U \Sigma_1 V_1^T \qquad U \Sigma V^T$

$V_1^T = \Sigma_1^{-1} U^T A$

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