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Example

$$A = \begin{vmatrix} 1 & 0 & 2 & 0 \\ 0 & 0 & -2 & 1 \end{vmatrix}$$

Singular values?

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compute: $A A^T =$

$$A A^T = \begin{vmatrix} 5 & -4 \\ -4 & 5 \end{vmatrix}$$

$\det(A A^T - \lambda I) = (5-\lambda)^2 - 16 \implies \lambda = 5 \pm 4$

$\sigma_1 = 3$
$\sigma_2 = 1$

$\lambda=9 \implies$
$$AA^T - \lambda I = \begin{vmatrix} -4 & -4 \\ -4 & -4 \end{vmatrix}$$

$$u_1 = \begin{vmatrix} 1 \\ -1 \end{vmatrix} / \sqrt{2}$$

$$u_2 = \begin{vmatrix} 1 \\ 1 \end{vmatrix} / \sqrt{2}$$

$$U = [u_1\ u_2] = \begin{vmatrix} 1 & 1 \\ -1 & 1 \end{vmatrix} / \sqrt{[2]}$$

$A = U \Sigma_1 V_1^T \implies$

$V_1^T = \Sigma_1^{-1} U^T A$

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Practice ex. 09

(a)

$$\min_{\{B \mid \mathrm{rank}(B) = k\}} \| A - B \| = \sigma_{k+1}$$

Qu: $\mathrm{rank}(B) \le k \rightarrow \| A - B \| \ge \sigma_{k+1}$

let $l=\mathrm{rank}(B) \le k \rightarrow \| A - B \| \ge \sigma_{l+1} \ge \sigma_{k+1}$

recall : $\sigma_1 \ge \ldots \sigma_k \ge \ldots$

(b) from (a) $\min [\ldots.] \ge \sigma_{k+1}$ – but

$\| A - A_k \| = \sigma_{k+1}$ .. $\implies$ min is equal
[achieved by A_k]

(c) start with
  $\mathrm{rank}(B) \le k \;\rightarrow\; \| A - B \| \ge \sigma_{k+1}$

  $\| A - B \| < \sigma_{k+1} \;\rightarrow\; \mathrm{rank}(B) > k$

  change k+1 into k →

 $\| A - B \| < \sigma_{k} \;\rightarrow\; \mathrm{rank}(B) > k-1$

 $\| A - B \| < \sigma_{k} \;\rightarrow\; \mathrm{rank}(B) \ge k$
□

(d) when k=r

   $\| A - B \| < \sigma_{r} \;\rightarrow\; \mathrm{rank}(B) \ge r = \mathrm{rank}(A)$

[a consequence: if A is full rank and B is close enough then B is of full rank]

(e) Q: let ε be given . and let k == the number of singular values that are > ε

   if $\| A - B \| \le \varepsilon$ then $\mathrm{rank}(B) \ge k$

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                  σ_{k+1}      σ_k               σ_1
___________________|________|______________|_____
                      |
                  ε<   σ_k
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we are in situation (c) ==> rank(B) ≥ k

Consider all matrices B such that $\|A - B\| \le \varepsilon$

their ranks must be ≥ k

the min of all these ranks is k.. it is achieved by A_k

== this is the definition or numerical rank or ε-rank of A

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