

=====
Example

$$A = \begin{pmatrix} 1 & 0 & 2 & 0 \\ 0 & 0 & -2 & 1 \end{pmatrix}$$

Singular values?

=====
compute: $A A^T =$

$$A A^T = \begin{pmatrix} 5 & -4 \\ -4 & 5 \end{pmatrix}$$

$$\det(A A^T - \lambda I) = (5-\lambda)^2 - 16 \implies \lambda = 5 \pm 4$$

$$\sigma_1 = 3$$

$$\sigma_2 = 1$$

$$\lambda=9 \implies A A^T - \lambda I = \begin{pmatrix} -4 & -4 \\ -4 & -4 \end{pmatrix}$$

$$u_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$u_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$U = [u_1 \ u_2] = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$$

$$A = U \Sigma_1 V_1^T \implies$$

$$V_1^T = \Sigma_1^{-1} U^T A$$

=====
Practice ex. 09

(a)

$$\min_{\{B \mid \text{rank}(B) = k\}} \|A - B\| = \sigma_{k+1}$$

$$\text{Qu: } \text{rank}(B) \leq k \rightarrow \|A - B\| \geq \sigma_{k+1}$$

$$\text{let } l = \text{rank}(B) \leq k \rightarrow \|A - B\| \geq \sigma_{l+1} \geq \sigma_{k+1}$$

$$\text{recall : } \sigma_1 \geq \dots \geq \sigma_k \geq \dots$$

(b) from (a) $\min [\dots] \geq \sigma_{k+1}$ - but

$$\|A - A_k\| = \sigma_{k+1} \dots \implies \min [\text{achieved by } A_k] \text{ is equal}$$

(c) start with
 $\text{rank}(B) \leq k \rightarrow \|A - B\| \geq \sigma_{k+1}$

$\|A - B\| < \sigma_{k+1} \rightarrow \text{rank}(B) > k$

change $k+1$ into $k \rightarrow$

$\|A - B\| < \sigma_k \rightarrow \text{rank}(B) > k-1$

$\|A - B\| < \sigma_k \rightarrow \text{rank}(B) \geq k$

□

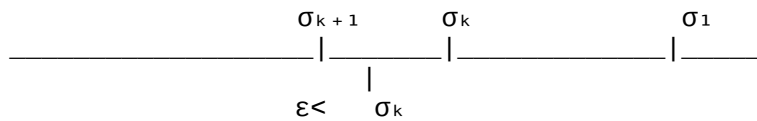
(d) when $k=r$

$\|A - B\| < \sigma_r \rightarrow \text{rank}(B) \geq r = \text{rank}(A)$

[a consequence: if A is full rank and B is close enough then B is of full rank]

(e) Q: let ε be given . and let k == the number of singular values that are $> \varepsilon$

if $\|A - B\| \leq \varepsilon$ then $\text{rank}(B) \geq k$



we are in situation (c) $\implies \text{rank}(B) \geq k$

Consider all matrices B such that $\|A - B\| \leq \varepsilon$

their ranks must be $\geq k$

the min of all these ranks is k .. it is achieved by A_k

\implies this is the definition or numerical rank or ε -rank of A

=====