

ON QUIZ # 5

Recall:

Let $A \in \mathbb{R}^{m \times n}$

$X = \text{Ran}(A) \implies$

$\mathbb{R}^m = X \oplus X^\perp \implies$

$\mathbb{R}^m = \text{Ran}(A) \oplus \text{Null}(A^T)$

$\mathbb{R}^n = \text{Ran}(A^T) \oplus \text{Null}(A)$

$\text{rank} = \dim(\text{Ran}(A)) = \dim(\text{Ran}(A^T))$

projector satisfies $P^2 = P$

householder transf. satisfies $P^2 = I$

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$$R = \begin{pmatrix} | & 1 & 0 & 0 & -1 & 0 & | \\ | & 0 & 0 & 1 & -3 & 0 & | \\ | & 0 & 0 & 0 & 1 & 1 & | \\ | & 0 & 0 & 0 & 2 & 1 & | \\ | & 0 & 0 & 0 & 0 & 0 & | \end{pmatrix} \quad \text{rank}(R)=2 \text{ ??}$$

$$R = \begin{pmatrix} | & x & x & x & | \\ | & 0 & x & x & | \\ | & 0 & 0 & 0 & | \\ | & 0 & 0 & 0 & | \end{pmatrix}$$

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$$\|A\|_2 = \max \|Ax\|_2 / \|x\|_2 = \max \|UAx\|_2 / \|x\|_2 = \|UA\|_2$$

$$\|A\|_F = \text{trace}(A^T A)^{1/2} = \text{trace}(A^T U^T U A)^{1/2} = \text{same}$$

$$\|UA\|_1 = \|A\|_1 \leq \text{incorrect}$$

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$$x = \begin{pmatrix} | & -1 & | \\ | & 0 & | \\ | & -2 & | \\ | & 2 & | \end{pmatrix} \quad \|x\| = 3$$

$$(I - 2ww^T)x = \alpha e_1$$

$$I - \beta vv^T$$

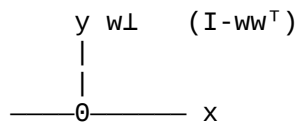
$$\alpha = \|x\| \implies v = x - \alpha e_1 = [-4, 0, -2, 2]^T \implies w = \dots$$

$$\alpha = -\|x\| \implies$$

$$v = x - \alpha e_1 = [2, 0, -2, 2]^T \implies w = \dots$$

(c) take $w = x/\|x\| \implies$

$$(I - ww^T)x = x - (x/\|x\|) x^T x / \|x\| = 0$$



$w^T y$ = projection of y onto x

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$$A = U \Sigma V^T \iff U \text{ } m \times m, \quad V \text{ } n \times n \quad \Sigma \text{ } m \times n \text{ [same as } A]$$

$$\Sigma = \begin{bmatrix} \Sigma_1 & 0 \\ 0 & 0 \end{bmatrix} \quad \Sigma_1 \text{ is } r \times r$$

$$U = [U_1, U_2] \quad V = [V_1, V_2]$$

$$A = U_1 \Sigma_1 V_1^T \iff \text{thin SVD}$$

$$= \sigma_1 u_1 v_1^T + \dots + \sigma_r u_r v_r^T$$

Pseudo-inverse:

$$A^+ = V_1 \Sigma_1^{-1} U_1^T \quad \text{same shape as } A^T$$

$$= (1/\sigma_1) v_1 u_1^T + \dots + (1/\sigma_r) v_r u_r^T$$

row dimension of $A^+ = ?$ n

col. dimension of $A^+ = ?$ m

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If A is $n \times n$ and invertible then

$A^+ =$ inverse of A

[$A = U \Sigma V^T$ where $r = n \implies U, V, \Sigma$ are invertible]

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General solution of LS problem.

$$A = U \Sigma V^T \quad \text{not necessarily full rank}$$

- 1) find all solutions
- 2) find solution (s) with smallest 2-norm.

$$Ax - b = U \Sigma V^T x - b$$

express solution in V basis: $x = V y$ y has n components

$$Ax - b = U \Sigma y - b = U[\Sigma y - U^T b]$$

$$\|Ax - b\| = \|U[\Sigma y - U^T b]\| = \|\Sigma y - U^T b\|$$

$$\Sigma = \begin{bmatrix} \Sigma_1 & 0 \\ 0 & 0 \end{bmatrix} \quad \Sigma_1 \text{ is } r \times r$$

$$y = [y_1, y_2]^T \quad V = [V_1, V_2], \quad U = [U_1, U_2]$$

$$\Sigma y = \begin{bmatrix} \Sigma_1 y_1 \\ 0 \end{bmatrix}$$

$$U^T b = \begin{bmatrix} U_1^T b \\ U_2^T b \end{bmatrix}$$

$$\|Ax - b\|^2 = \left\| \begin{bmatrix} \Sigma_1 y_1 - U_1^T b \\ -U_2^T b \end{bmatrix} \right\|^2$$

Minimum achieved when 1st part is zero
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Solution (s) with smallest 2-norm

$$\|x\|^2 = \|A^+ b\|^2 + \|w\|^2$$

smallest when $w = 0$ i.e., when $y_2 = 0$

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$$A^+ = V_1 \Sigma_1^{-1} U_1^T$$

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$$A = U_1 \Sigma_1 V_1^T$$

$$A^+ A = V_1 \Sigma_1^{-1} U_1^T U_1 \Sigma_1 V_1^T = V_1 V_1^T = \text{Orthogonal Projector}$$

[P= U U^T where U^T U=I]

$$A A^+ = U_1 U_1^T$$