ON QUIZ # 5

Recall:

Let $A \in \mathbb{R}^{m \times n}$

$X = \mathrm{Ran}(A)$ ==>

$\mathbb{R}^{m} = X \oplus X^{\perp}$ ===>

$\mathbb{R}^{m} = \mathrm{Ran}(A) \oplus \mathrm{Null}(A^T)$

$\mathbb{R}^{n} = \mathrm{Ran}(A^T) \oplus \mathrm{Null}(A)$

rank = $\dim(\mathrm{Ran}(A)) = \dim(\mathrm{Ran}(A^T))$

projector satisfies $P^2 = P$
householder transf. satisfies $P^2 = I$

==

$$R = \begin{vmatrix} 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & -3 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{vmatrix} \qquad \mathrm{rank}(R)=2\ ??$$

$$R = \begin{vmatrix} x & x & x \\ 0 & x & x \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{vmatrix}$$

==

$$\| A \|_2 = \max \| Ax \|_2 / \| x\|_2$$
$$= \max \| U Ax \|_2 / \| x\|_2 = \| U A\|_2$$

$$\| A\|_F = \mathrm{trace}\,(A^T A)^{1/2}$$
$$= \mathrm{trace}\,(A^T U^T U A)^{1/2} = \text{same}$$

$\| U A \|_1 = \| A \|_1$ <== incorrect

==

$$x == \begin{vmatrix} -1 \\ 0 \\ -2 \\ 2 \end{vmatrix} \qquad \| x \| = 3$$

$(I-2\, w\, w^T)\, x = \alpha\, e_1$

$I-\beta\, v\, v^T$

$\alpha = \| x \|$ ==>
$v = x - \alpha e_1 = [-4, 0, -2, 2]^T$ ==> w = ..

$\alpha = - \| x \|$ ==>

$v = x - \alpha e_1 = [2, 0, -2, 2]^T$ ==> w = ..

(c) take $w = x/\|x\|$ ==>

$(I-ww^T)x = x - (x/\|x\|)\, x^T x /\|x\| = 0$

==

```
    y w⊥   (I-wwᵀ)
    |
    |
————0——————— x
```

$w^T y$ = projection of y onto x

==

$A = U \Sigma V^T$ <=== U mxm, V nxn Σ mxn [same as A]

$$\Sigma = \begin{vmatrix} \Sigma_1 & 0 \\ 0 & 0 \end{vmatrix}$$ Σ_1 is r x r

$U =[U_1, U_2]$ $V =[V_1, V_2]$

$A = U_1 \Sigma_1 V_1^T$ <== thin SVD

$= \sigma_1 u_1 v_1^T + \cdots \sigma_r u_r v_r^T$

Pseudo-inverse:

$A^+ = V_1 \Sigma_1^{-1} U_1^T$ same shape as A^T

$= (1/\sigma_1)\, v_1 u_1^T + \cdots (1/\sigma_r)\, v_r u_r^T$

row dimension of A^+ = ? n
col. dimension of A^+ = ? m

==

If A is nxn and invertible then

A^+ = inverse of A

[$A = U \Sigma V^T$ where r = n == U, V, Σ are invertible]

==

General solution of LS problem.

$A = U \Sigma V^T$ not necessarily full rank

1) find all solutions
2) find solution (s) with smallest 2-norm.

$Ax - b = U \Sigma V^T x - b$

express solution in V basis: $x = V y$ y has n components

$Ax - b = U \Sigma y - b = U[\Sigma y - U^T b]$

$\|Ax - b\| = \| U[\Sigma y - U^T b] \| = \| \Sigma y - U^T b \|$

$$\Sigma = \begin{vmatrix} \Sigma_1 & 0 \\ 0 & 0 \end{vmatrix} \quad \Sigma_1 \text{ is } r \times r$$

$y = [y_1, y_2]^T$ $V = [V_1, V_2]$, $U = [U_1, U_2]$

$$\Sigma y = \begin{vmatrix} \Sigma_1 y_1 \\ 0 \end{vmatrix}$$

$$U^T b = \begin{vmatrix} U_1^T b \\ U_2^T b \end{vmatrix}$$

$$\| Ax-b \|^2 = \left\| \begin{matrix} \Sigma_1 y_1 - U_1^T b \\ -U_2^T b \end{matrix} \right\|^2$$

Minimum achieved when 1st part is zero
==

Solution (s) with smallest 2-norm

$||x||^2 = || A^+ b||^2 + || w ||^2$

smallest when w == 0 i.e., when $y_2 = 0$

==

$A^+ = V_1 \Sigma_1^{-1} U_1^T$
================

$A = U_1 \Sigma_1 V_1^T$

$A^+ A = V_1 \Sigma_1^{-1} U_1^T U_1 \Sigma_1 V_1^T =$
$V_1 V_1^T$ = Orthogonal Projector
[$P = U U^T$ where $U^T U = I$]

$A A^+ = U_1 U_1^T$