1. T/F: SVD of a matrix always exists

2. T/F: Number of nonzero singular values of A = rank of A

3. Terminology: Left singular vectors, right singular vectors
U V

$$A = U \Sigma V^T$$

$A \; v_i = \sigma_i u_i$

$A^T u_i = \sigma_i v_i$ ==> $u_i^T A = \sigma_i v_i^T$

4. Computing sing. values & vectors
– nonzero sing. values of a 2x5 matrix A?

$A (A^T u_i) = A (\sigma_i v_i) = \sigma_i^2 u_i$ ==> can get U

To get V: Use relation $A = U \Sigma_1 V_1^T \rightarrow V_1^T = \Sigma_1^{-1} U^T A$

5. If you have a pair σ_i v_i how would you compute u_i?

If you have a pair σ_1, u_1 how would you compute v_1?
$u_1^T A / \sigma_1 = v_1^T$

$A^T u_i = \sigma_i v_i$ => $v_i = (A^T u_i)/\sigma_i$

6. A with nonzero singular values $\sigma_1 \cdots \sigma_r$ [r = rank]
B a matrix of rank = 1

what can you say about $\| A - B \|_2 \geq \sigma_2$

What about $\min_{rank(B)=1} \| A - B \|_2 = \sigma_2$

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Eigenvalues of

$$\begin{vmatrix} 1 & 2 & -4 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{vmatrix}$$

$$\det(A - t I) = \begin{vmatrix} 1-t & 2 & -4 \\ 0 & 1-t & 2 \\ 0 & 0 & 2-t \end{vmatrix} = (1-t)^2 (2-t)$$

$\Lambda = \{1, 1, 2\}$ 1 = double eigenvalue

1 with alg. multiplicity of 2 [double] and
2 with alg. multiplicity of 1 [simple]

Geometric multiplicities

For $\lambda = 1$. eigenvectors?

$$\begin{vmatrix} 0 & 2 & -4 \\ 0 & 0 & 2 \\ 0 & 0 & 1 \end{vmatrix} \begin{matrix} x_1 \\ x_2 \\ x_3 \end{matrix} = 0$$

$x_3 = 0$; $x_3 = 0$; $2 x_2 = 0 \Rightarrow x_2 = 0$

eigenvector = $[1\ 0\ 0]^T$
Only one eigenvector == geom. mult. = 1   [→ not semi-simple]

For $\lambda = 2$: simple eigenvalue
alg. mult. = 1, geom. mult. = 1

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ex 2.   replace a_{33} by one

$$\det(A - t I) = \begin{vmatrix} 1-t & 2 & -4 \\ 0 & 1-t & 2 \\ 0 & 0 & 1-t \end{vmatrix} = (1-t)^3$$

One eigenvalue $\lambda=1$ of alg. mult. = 3.

eigenvectors:

$$\begin{vmatrix} 0 & 2 & -4 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{vmatrix} \begin{matrix} x_1 \\ x_2 \\ x_3 \end{matrix} = 0$$

we get the same result. $x = [1, 0, 0]^T$

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ex 3.   replace a_{33} by one and a_{12} by 0

$$\det(A - t I) = \begin{vmatrix} 1-t & 0 & -4 \\ 0 & 1-t & 2 \\ 0 & 0 & 1-t \end{vmatrix} = (1-t)^3$$

One eigenvalue $\lambda=1$ of alg. mult. = 3.

eigenvectors:

$$\begin{vmatrix} 0 & 0 & -4 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{vmatrix} \begin{matrix} x_1 \\ x_2 \\ x_3 \end{matrix} = 0$$

$x_3 = 0$ is the only condition.

$x = [1, 0, 0]^T$ and $x = [0, 1, 0]^T$

also
$x = [1, 1, 0]^T$ and $x = [1, -1\ 0]^T$

Alg. mult. = 3, geom. mult = 2.
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Theorem A diagonalizable ⇔ A has n lin. independent eigenvectors

A diagonalizable ⇔ $A = U \Lambda U^{-1}$
A diagonalizable ⇔ $A U = U \Lambda$
⇔ $A [u_1, \ldots , u_n] = [u_1, \ldots , u_n] \, Diag(\lambda_1, \ldots, \lambda_n)$
⇔ $A [u_1, \ldots , u_n] = [\lambda_1 u_1, \ldots, \lambda_n u_n]$
⇔ $[Au_1, \ldots , Au_n] = [\lambda_1 u_1, \ldots, \lambda_n u_n]$

⇔ $A u_i = \lambda_i u_i$ for $i=1, \ldots, n$
and the u_i's are lin. independent [columns of an invertible matrix.]

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Schur form theorem

ex. 6 show

$$P A P^H = \begin{vmatrix} \lambda & * \\ 0 & A_2 \end{vmatrix}$$

$P A P^H e_1 = P A x = P \lambda x = \lambda Px = \lambda e_1$ Done
$== x$

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If A is hermitian:

$Q R Q^H = Q R^H Q^H \rightarrow R = R^H$

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