$\|A u\| = \|\lambda u\| = |\lambda| \; \|u\|$

$\|A u\| / \|u\| = |\lambda| \le \max \|Ax\|/\|x\| = \|A\|$

===

Gerschgorin's theorem

Disk $D(c, r)$ = disk centered at c with radius r
$= \{ z \in C \mid |z-c| \le r \}$

theorem says that any λ belongs to one of the n disks

$D(a_{ii}, \rho_i)$ – where $\rho_i = \sum_{j \ne i} |a_{ij}|$

$\lambda \in \bigcup D(a_{ii}, \rho_i)$

===

Proof of Gerschgorin's theorem

negation of theorem is

$\exists \; \lambda$ s.t. for all i $| \lambda - a_{ii} | > \rho_i$ (**)
[————————-> $D-\lambda I$ is invertible]

$D-\lambda I$ is invertible .. therefore

$A - \lambda I = D-\lambda I -(D-A) = (D-\lambda I) [I - (D-\lambda I)^{-1}(D-A)]$

$= (D-\lambda I) [I - (D-\lambda I)^{-1}F]$ (&&)
============================

Diagonal entries of F: they are all zeros.
Off-diago. entries in row i : $-a_{ij}$
1-norm of row i = $\rho_i = \sum_{j \ne i} |a_{ij}|$

Entries of row i of $(D-\lambda I)^{-1} F$:
diag entry is 0
off-diag. $-a_{ij}$ /

$$\text{inv}(D - \lambda I) = \begin{vmatrix} 1/(a_{11}-\lambda) & 0 & \dots & \\ & 1/(a_{22}-\lambda) & 0 & \dots \\ & & & \\ & & & 1/(a_{nn}-\lambda) \end{vmatrix}$$

Row i of $(D-\lambda I)^{-1} F = 1/(a_{ii}-\lambda) \times \text{row}_i$ of F

$\| \text{Row } i \|_1 = \rho_i / |a_{ii}-\lambda| < 1$ because of (**)

$\| (D-\lambda I)^{-1} F \|_\infty < 1$

Then (&&) implies that $A-\lambda I$ is invertible (of the form

$(D-\lambda I)^{-1}(I-E)$ where $||E||<1$)
CONTRADICTION. □

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