UNIVERSITY OF MINNESOTA TWIN CITIES

C S C I 5304 **Fall 2021**

COMPUTATIONAL ASPECTS OF MATRIX THEORY

Class time	:	*MW 4:00 – 5:15 pm*
Room	:	*Keller 3-230 or Online*
Instructor	:	*Daniel Boley*

Lecture notes: http://www-users.cselabs.umn.edu/classes/Fall-2021/csci5304/

August 27, 2021

A few applications of the SVD

Many methods require to approximate the original data (matrix) by a low rank matrix before attempting to solve the original problem

➤ Regularization methods require the solution of a least-squares linear system $Ax = b$ approximately in the dominant singular space of A

➤ The Latent Semantic Indexing (LSI) method in information retrieval, performs the "query" in the dominant singular space of A

➤ Methods utilizing Principal Component Analysis, e.g. Face Recognition.

Commonality: Approximate A (or $A^\dagger$) by a lower rank approximation A_k (using dominant singular space) before solving original problem.

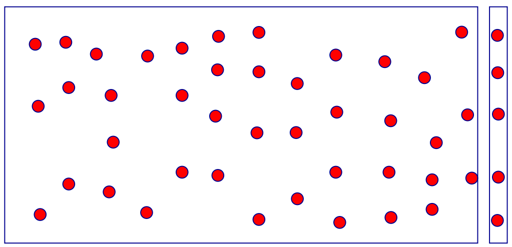
➤ This approximation captures the main features of the data while getting rid of noise and redundancy

Note: Common misconception: 'we need to reduce dimension in order to reduce computational cost'. In reality: using less information often yields better results. This is the problem of overfitting.

➤ Good illustration: Information Retrieval (IR)

Information Retrieval: Vector Space Model

➤ Given: a collection of documents (columns of a matrix A) and a query vector q.



➤ Collection represented by an $m \times n$ term by document matrix with $\boxed{a_{ij} = L_{ij} G_i N_j}$

➤ Queries ('pseudo-documents') q are represented similarly to a column

Vector Space Model - continued

- Problem: find a column of A that best matches q
- Similarity metric: angle between the column and q - Use cosines:

$$\frac{|c^T q|}{\|c\|_2 \|q\|_2}$$

- To rank all documents we need to compute

$$s = A^T q$$

- s = similarity vector.
- Literal matching – not very effective.

Use of the SVD

- Many problems with literal matching: *polysemy*, *synonymy*, ...
- Need to extract intrinsic information – or underlying "semantic" information –
- Solution (LSI): replace matrix A by a low rank approximation using the Singular Value Decomposition (SVD)

$$A = U\Sigma V^T \quad \to \quad A_k = U_k \Sigma_k V_k^T$$

- U_k : term space, V_k: document space.
- Refer to this as Truncated SVD (TSVD) approach

New similarity vector:

$$s_k = A_k^T q = V_k \Sigma_k U_k^T q$$

Issues:

- Problem 1: How to select k?
- Problem 2: computational cost (memory + computation)
- Problem 3: updates [e.g. google data changes all the time]
- Not practical for very large sets

LSI : an example

```
%% D1   :  INFANT & TODDLER first aid
%% D2   :  BABIES & CHILDREN's room for your HOME
%% D3   :  CHILD SAFETY at HOME
%% D4   :  Your BABY's HEALTH and SAFETY
%%      :  From INFANT to TODDLER
%% D5   :  BABY PROOFING basics
%% D6   :  Your GUIDE to easy rust PROOFING
%% D7   :  Beanie BABIES collector's GUIDE
%% D8   :  SAFETY GUIDE for CHILD PROOFING your HOME
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%% TERMS: 1:BABY   2:CHILD 3:GUIDE 4:HEALTH 5:HOME
%%         6:INFANT 7:PROOFING  8:SAFETY 9:TODDLER
%% Source: Berry and Browne, SIAM., '99
```

- Number of documents: 8
- Number of terms: 9

➤ Raw matrix (before scaling).

$A =$

d1	d2	d3	d4	d5	d6	d7	d8	
	1		1	1		1		bab
	1	1					1	chi
					1	1	1	gui
			1					hea
	1	1					1	hom
1			1					inf
				1	1		1	pro
		1	1				1	saf
1			1					tod

✍D1 Get the anwser to the query `Child Safety`, so

$$q = [0\ 1\ 0\ 0\ 0\ 0\ 0\ 1\ 0]$$

using cosines and then using LSI with $k = 3$.

Dimension reduction

Dimensionality Reduction (DR) techniques pervasive to many applications

➤ Often main goal of dimension reduction is not to reduce computational cost. Instead:

- Dimension reduction used to reduce noise and redundancy in data
- Dimension reduction used to discover patterns (e.g., supervised learning)

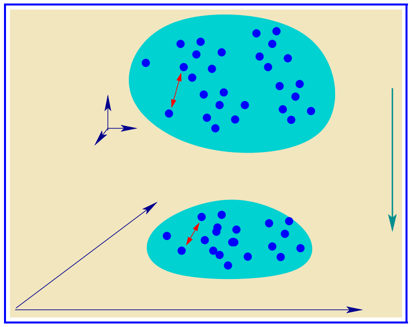
➤ Techniques depend on desirable features or application: Preserve angles? Preserve distances? Maximize variance? ..

The problem

➤ Given $d \ll m$ find a mapping $\Phi : x \in \mathbb{R}^m \longrightarrow y \in \mathbb{R}^d$

➤ Mapping may be explicit (e.g., $y = V^T x$)

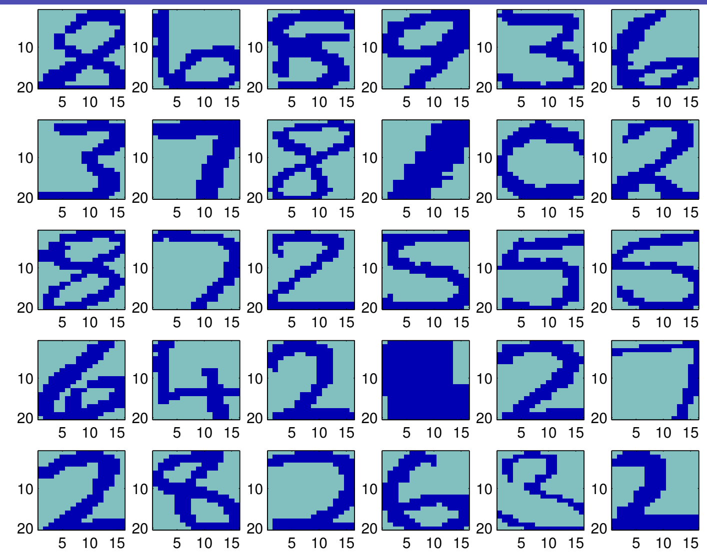
➤ Or implicit (nonlinear)



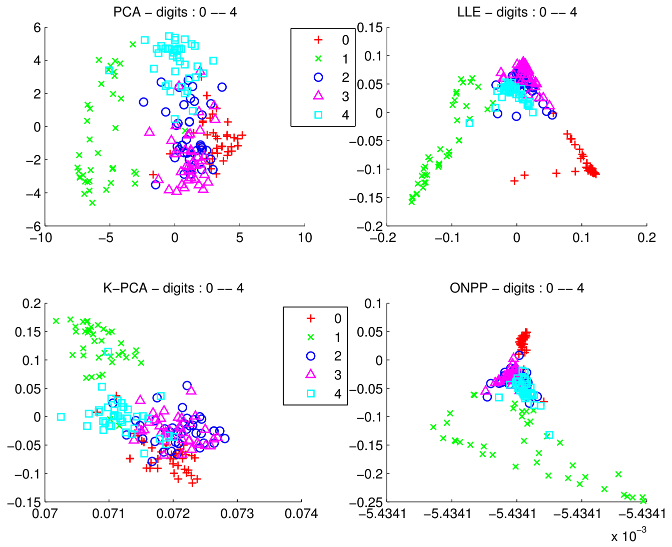
Practically: Find a low-dimensional representation $Y \in \mathbb{R}^{d \times n}$ of $X \in \mathbb{R}^{m \times n}$.

➤ Two classes of methods: (1) projection techniques and (2) nonlinear implicit methods.

Example: Digit images (a sample of 30)



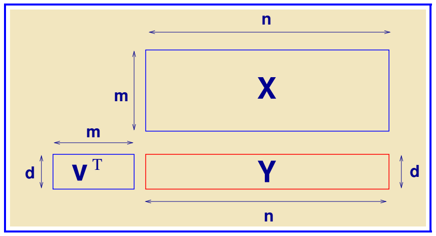
A few 2-D 'reductions':



Projection-based Dimensionality Reduction

Given: a data set $X = [x_1, x_2, \ldots, x_n]$, and d the dimension of the desired reduced space Y.

Want: a linear transformation from X to Y



$X \in \mathbb{R}^{m\times n}$
$V \in \mathbb{R}^{m\times d}$
$Y = V^\top X$
$\rightarrow \quad Y \in \mathbb{R}^{d\times n}$

- m-dimens. objects (x_i) 'flattened' to d-dimens. space (y_i)

Problem: Find the best such mapping (optimization) given that the y_i's must satisfy certain constraints

Principal Component Analysis (PCA)

- PCA: find V (orthogonal) so that projected data $Y = V^T X$ has maximum variance
- Maximize over all orthogonal $m \times d$ matrices V:

$$\sum_i \left\| y_i - \frac{1}{n}\sum_j y_j \right\|_2^2 = \cdots = \text{Tr}\left[V^\top \bar{X}\bar{X}^\top V\right]$$

Where: $\bar{X} = [\bar{x}_1, \cdots, \bar{x}_n]$ with $\bar{x}_i = x_i - \mu$, μ = mean.

Solution: $V = \{$ dominant eigenvectors $\}$ of covariance matrix

- i.e., Optimal V = Set of left singular vectors of $\bar{X}$ associated with d largest singular values.

✍2 Show that $\bar{X} = X(I - \frac{1}{n}ee^T)$ (here e = vector of all ones). What does the projector $(I - \frac{1}{n}ee^T)$ do?

✍3 Show that solution V also minimizes 'reconstruction error' ..

$$\sum_i \|\bar{x}_i - VV^T\bar{x}_i\|^2 = \sum_i \|\bar{x}_i - V\bar{y}_i\|^2$$

✍4 .. and that it also maximizes $\sum_{i,j} \|y_i - y_j\|^2$

Matrix Completion Problem

Consider a table of movie ratings. You want to predict missing ratings by assuming commonality (low rank matrix).

	given data			predictions		
movie	Paul	Jane	Ann	Paul	Jane	Ann
Title-1	−1	3	−1	−1.2	1.7	−0.7
Title-2	4	x	3	2.8	−1.2	2.5
Title-3	−3	1	−4	−2.7	1.0	−2.5
Title-4	x	−1	−1	−0.5	−0.3	−0.6
Title-5	3	−2	1	1.8	−1.4	1.4
Title-6	−2	3	x	−1.6	1.8	−1.2
	$\boldsymbol{A}$			$\boldsymbol{X}$		

➤ Minimize $\|(\boldsymbol{X} - \boldsymbol{A})_{\text{mask}}\|_{\boldsymbol{F}}^2 + \boldsymbol{\mu}\|\boldsymbol{X}\|_*$

"minimize sum-of-squares of deviations from known ratings plus sum of singular values of solution (to reduce the rank)."