UNIVERSITY OF MINNESOTA TWIN CITIES

C S C I 5304 **Fall 2021**

COMPUTATIONAL ASPECTS OF MATRIX THEORY

Class time	:	*MW 4:00 – 5:15 pm*
Room	:	*Keller 3-230 or Online*
Instructor	:	*Daniel Boley*

Lecture notes: http://www-users.cselabs.umn.edu/classes/Fall-2021/csci5304/

August 27, 2021

APPLICATION: GRAPH PARTITIONING

Graph Laplacians - Definition

➤ "Laplace-type" matrices associated with general undirected graphs – useful in many applications

➤ Given a graph $G = (V, E)$ define

- A matrix W of weights w_{ij} for each edge
- Assume $w_{ij} \geq 0,$, $w_{ii} = 0$, and $w_{ij} = w_{ji}\ \forall(i, j)$
- The diagonal matrix $D = diag(d_i)$ with $d_i = \sum_{j \neq i} w_{ij}$

➤ Corresponding *graph Laplacian* of G is:

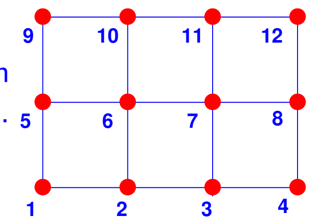
$$L = D - W$$

➤ Gershgorin's theorem $\rightarrow$ L is positive semidefinite

➤ Simplest case:

$$w_{ij} = \begin{cases} 1 \text{ if } (i, j) \in E \& i \neq j \\ 0 \quad \text{else} \end{cases} \qquad D = \text{diag}\left[d_i = \sum_{j \neq i} w_{ij}\right]$$

✍1 Define the graph Laplacian for the graph associated with the simple mesh shown next. [use the simple weights of 0 or 1]



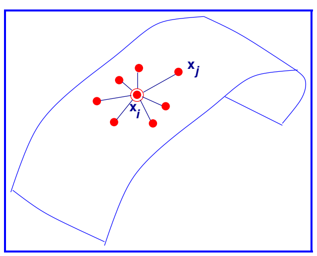
➤ $\text{Adjacency matrix} = W = \begin{bmatrix} . & 1 & & & 1 & & & & & & & \\ 1 & . & 1 & & & 1 & & & & & & \\ & 1 & . & 1 & & & 1 & & & & & \\ & & 1 & . & & & & 1 & & & & \\ 1 & & & & . & 1 & & & 1 & & & \\ & 1 & & & 1 & . & 1 & & & 1 & & \\ & & 1 & & & 1 & . & 1 & & & 1 & \\ & & & 1 & & & 1 & . & & & & 1 \\ & & & & 1 & & & & . & 1 & & \\ & & & & & 1 & & & 1 & . & 1 & \\ & & & & & & 1 & & & 1 & . & 1 \\ & & & & & & & 1 & & & 1 & . \end{bmatrix}$; $\text{Incidence matrix} = N = \begin{bmatrix} -1 & . & . & . & +1 & . & . & . & . & . & . & . \\ -1 & +1 & . & . & . & . & . & . & . & . & . & . \\ . & -1 & . & . & . & +1 & . & . & . & . & . & . \\ . & -1 & +1 & . & . & . & . & . & . & . & . & . \\ . & . & -1 & . & . & . & +1 & . & . & . & . & . \\ . & . & -1 & +1 & . & . & . & . & . & . & . & . \\ . & . & . & -1 & . & . & . & +1 & . & . & . & . \\ . & . & . & . & -1 & . & . & . & +1 & . & . & . \\ . & . & . & . & -1 & +1 & . & . & . & . & . & . \\ . & . & . & . & . & . & -1 & . & . & . & +1 & . \\ . & . & . & . & . & -1 & . & . & . & +1 & . & . \\ . & . & . & . & . & -1 & +1 & . & . & . & . & . \\ . & . & . & . & . & . & -1 & +1 & . & . & . & . \\ . & . & . & . & . & . & . & -1 & . & . & . & +1 \\ . & . & . & . & . & . & . & . & -1 & +1 & . & . \\ . & . & . & . & . & . & . & . & . & -1 & +1 & . \\ . & . & . & . & . & . & . & . & . & . & -1 & +1 \end{bmatrix}$;

$$L = D - W = N^T N$$

A few properties of graph Laplacians

✍₂ What is the difference with the discretization of the Laplace operator in 2-D for case when mesh is the same as this graph?

A few properties of graph Laplacians



Strong relation between $x^T L x$ and local distances between entries of x

➤ Let L = any matrix s.t. $L = D - W$, with $D = diag(d_i)$ and

$$w_{ij} \geq 0, \qquad d_i = \sum_{j \neq i} w_{ij}$$

Property 1: for any $x \in \mathbb{R}^n$:

$$x^\top L x = \frac{1}{2} \sum_{i,j} w_{ij} |x_i - x_j|^2$$

Property 2: (generalization) for any $Y \in \mathbb{R}^{d \times n}$:

$$\text{Tr}\,[Y L Y^\top] = \frac{1}{2} \sum_{i,j} w_{ij} \|y_i - y_j\|^2$$

Property 3: For the particular $L = I - \frac{1}{n}\mathbf{1}\mathbf{1}^\top$

$$XLX^\top = \bar{X}\bar{X}^\top == n \times \text{Covariance matrix}$$

Property 4: L is singular and admits the null vector e =`ones(n,1)`

Property 5: (Graph partitioning) Consider situation when $w_{ij} \in \{0, 1\}$. If x is a vector of signs (± 1) then

$$x^\top L x = 4 \times (\text{'number of edge cuts'})$$

edge-cut = pair (i, j) with $x_i \neq x_j$

- Would like to minimize (Lx, x) subject to $x \in \{-1, 1\}^n$ and $e^T x = 0$ [balanced sets]
- Wll solve a relaxed form of this problem

- Consider any symmetric (real) matrix A with eigenvalues $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$ and eigenvectors $u_1, \cdots, u_n$
- Recall that: (Min reached for $x = u_1$)

$$\min_{x \in \mathbb{R}^n} \frac{(Ax, x)}{(x, x)} = \lambda_1$$

- In addition: (Min reached for $x = u_2$)

$$\min_{x \perp u_1} \frac{(Ax, x)}{(x, x)} = \lambda_2$$

- For a graph Laplacian $u_1 = e$ = vector of all ones and
- ...vector u_2 is called the Fiedler vector. It solves a relaxed form of the problem -

$$\min_{x \in \{-1,1\}^n;\ e^T x = 0} \frac{(Lx, x)}{(x, x)} \rightarrow \min_{x \in \mathbb{R}^n;\ e^T x = 0} \frac{(Lx, x)}{(x, x)}$$

- Define $v = u_2$ then $lab = sign(v - med(v))$

Spectral Graph Partitioning

Idea:

- Partition graph in two using fiedler vectors
- Cut largest in two ..
- Repeat until number of desired partitions is reached
- Use the Lanczos algorithm to compute the Fiedler vector at each step

Application: Spectral Graph Partitioning

➤ Let N be the incidence matrix: $N_{ij} = \pm 1$ if i-th edge is incident on the j-th vertex.

➤ For example: A↔C,D, B↔D, C↔A, D↔A,B (undirected graph):

$$N = \begin{pmatrix} 1 & 0 & -1 & 0 \\ 1 & 0 & 0 & -1 \\ 0 & -1 & 0 & 1 \end{pmatrix},$$

yielding Laplacian = diagonal matrix of degrees — Adjacency matrix :

$$N^T N = L = \begin{pmatrix} 2 & 0 & -1 & -1 \\ 0 & 1 & 0 & -1 \\ -1 & 0 & 1 & 0 \\ -1 & -1 & 0 & 2 \end{pmatrix}.$$

Normalized Graph Cuts

Mark a partitioning of the vertices: $n_- = 1,\ n_+ = 3$
$v = [1, 1, 1, -3]^T/\sqrt{3 \cdot 1} = [n_-, n_-, n_-, -n_+]^T/\sqrt{n_- n_+}$.

Then

$$\frac{v^T L v}{v^T v} = |\text{cut}| \cdot \left(\frac{1}{n_-} + \frac{1}{n_+}\right) \quad \text{and:}$$

$$v^T e = 0, \text{ where } e = [1, 1, 1, 1]^T = \text{eigenvector of } L.$$

➤ Approximately minimize this with an eigenvector of L:

```
-1.E-15 (.500000 .500000 .500000 .500000) ← 'null' vector
.585786 (-.27059 .653281 -.65328 .270598) ← 'Fiedler'
2.00000 (.500000 -.50000 -.50000 .500000)    vector
3.41421 (.653281 .270598 -.27059 -.65328)
```


Analogy with Electrical Networks

➤ Let 1 amp current is applied between nodes 1 and n.
Assume unit resistances on every link.
What is the voltage drop?

➤ Let v = vector of voltage levels at each node.
Ohm's Law: $Nv = \mathrm{i}$ = currents across every link.
Kirchoff's Law: $N^T \mathrm{i} = b$, where $b = (1, 0, \ldots, 0, -1)^T$.

➤ Solve $N^T N v = b$ for voltages. Use $L = N^T N$.
Try $v = L^\dagger b$

✍D3 Show $N^T N = L$ and $Lv = b$.

➤ Voltage drop from 1 to n is proportional to the average commute time for a random walk from 1 to n and back. This is a square of a metric distance between nodes.

Application: Google's Page rank

➤ Idea is to put order into the web by ranking pages by their importance..

➤ Install the google-toolbar on your laptop or computer

`http://toolbar.google.com/`

➤ Tells you how important a page is...

➤ Google uses this for searches..

➤ Updated regularly..

➤ Still a lot of mystery in what is in it..

Page-rank - explained

Main point: A page is important if it is pointed to by other important pages.

➤ Importance of your page (its PageRank) is determined by summing the page ranks of all pages which point to it.

➤ Weighting: If a page points to several other pages, then the weighting should be distributed proportionally.

➤ Imagine many tokens doing a random walk on this graph:

- (δ/n) chance to follow one of the n links on a page,
- $(1-\delta)$ chance to jump to a random page.
- What's the chance a token will land on each page?

➤ If `www.cs.umn.edu/~boley` points to 10 pages including yours, then you will get $1/10$ of the credit of my page.

Page-Rank - definitions

If T_1, ..., T_n point to page T_i then

$$\rho(T_i) \; = \; 1 - \delta + \delta \left[\frac{\rho(T_1)}{|T_1|} + \frac{\rho(T_2)}{|T_2|} + \cdots \frac{\rho(T_n)}{|T_n|} \right]$$

➤ $|T_j|$ = count of links going out of Page T_j. So the 'vote' $\rho(T_j)$ is spread evenly among $|T_j|$ links.

➤ Sum of all PageRanks == 1: $\Sigma_T \rho(T) = 1$

➤ δ is a 'damping' parameter close to 1 – e.g. 0.85

➤ Defines a (possibly huge) Hyperlink matrix H

$$h_{ij} = \begin{cases} \frac{1}{|T_i|} & \text{if } i \text{ points to } j \\ 0 & \text{otherwise} \end{cases}$$

✍D4 4 Nodes

A points to B and D

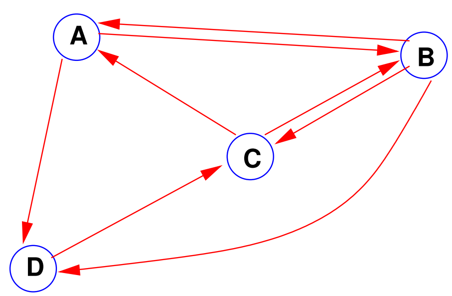
B points to A, C, and D

C points to A and B

D points to C

1) What is the H matrix?

2) the graph?



	A	B	C	D
A		$1/2$		$1/2$
B	$1/3$		$1/3$	$1/3$
C	$1/2$	$1/2$		
D			1	

➤ Row- sums of H are = 1.

➤ Sum of all PageRanks will be one: $$\sum_{\text{All-Pages}\,A} \rho(A) = 1.$$

➤ H is a stochastic matrix [actually it is forced to be by changing zero rows]

Algorithm (PageRank)

1. Select initial row vector v ($v \geq 0$)
2. For i=1:maxitr
3. $\quad v := (1-\delta)e^T + \delta v H$
4. end

✍15 Do a few steps of this algorithm for previous example with $\delta = 0.85$.

➤ This is a row iteration..

$$\boxed{\ v\ } = \boxed{(1-\delta)e^T} + \boxed{\ v\ } \cdot \boxed{\delta H}$$

A few properties:

➤ v will remain ≥ 0. [combines non-negative vectors]

➤ More general iteration is of the form

$$v := v\underbrace{[(1-\delta)E + \delta H]}_{G} \quad \textit{with} \quad E = ez^T$$

where z is a probability vector $e^T z = 1$ [Ex. $z = \frac{1}{n}e$]

➤ A variant of the power method.

➤ e is a right-eigenvector of G associated with $\lambda = 1$. We are interested in the left eigenvector.

Kleinberg's Hubs and Authorities

➤ Idea is to put order into the web by ranking pages by their degree of Authority or "Hubness".

➤ An Authority is a page pointed to by many important pages.
• Authority Weight = sum of Hub Weights from In-Links.

➤ A Hub is a page that points to many important pages:
• Hub Weight = sum of Authority Weights from Out-Links.

➤ Source:

`http://www.cs.cornell.edu/home/kleinber/auth.pdf`

Computation of Hubs and Authorities

➤ Simplify computation by forcing sum of squares of weights to be 1.

➤ $\text{Auth}_j = x_j = \sum_{i:(i,j)\in \text{Edges}} \text{Hub}_i$.

➤ $\text{Hub}_i = y_i = \sum_{j:(i,j)\in \text{Edges}} \text{Auth}_j$.

➤ Let A = Adjacency matrix: $a_{ij} = 1$ if $(i,j) \in \text{Edges}$.

➤ $\mathbf{y} = A\mathbf{x}$, $\mathbf{x} = A^T\mathbf{y}$.

➤ Iterate $\ldots$ to leading eigenvectors of A^TA & AA^T.

➤ Answer: Leading Singular Vectors!