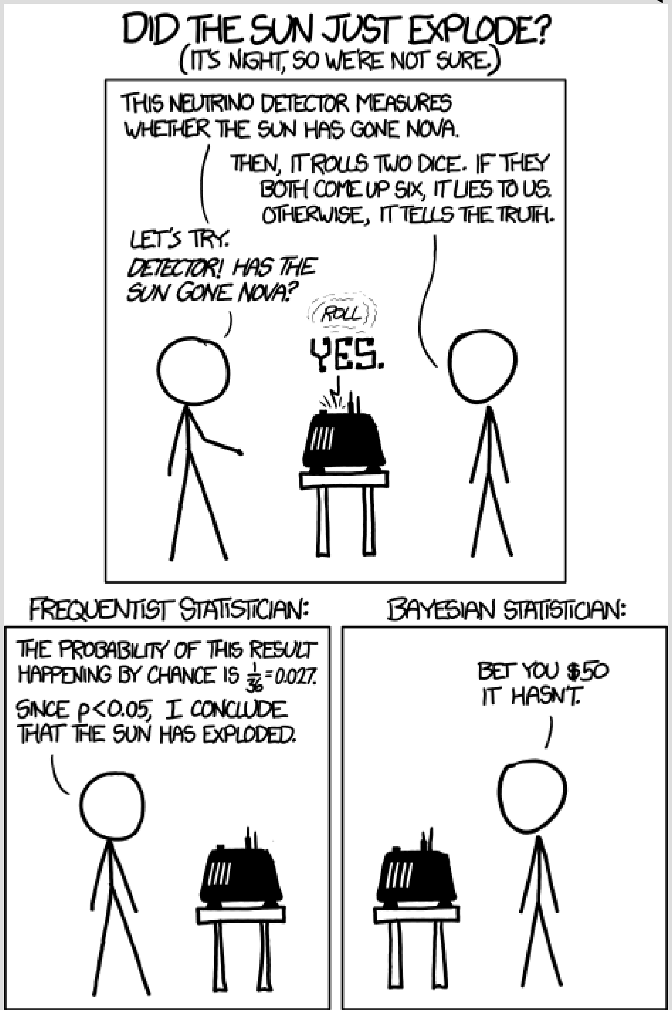
Exact inference (Ch. 14)



Bayesian Network

A Bayesian network (Bayes net) is:

(1) a directed graph

(2) acyclic

Additionally, Bayesian networks are assumed to be defined by conditional probability tables

(3) P(x | Parents(x))

We have actually used one of these before...

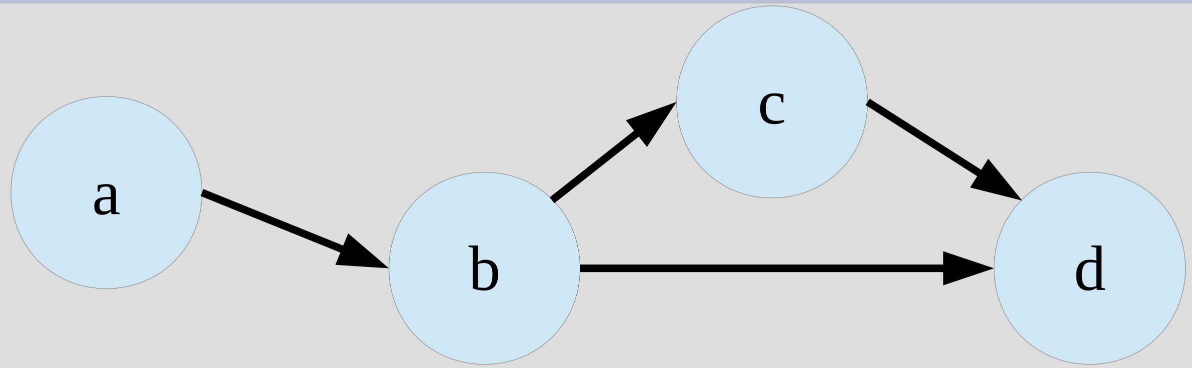
Bayesian Network

I have been lax on capitalization (e.g. $P(a)$ vs. $P(A)$), but not today

Capitalization = set of outcomes
Lower-case = a single outcome
(by letter, so “a” is an outcome of “A”)

So $P(A) = <P(a), P(\neg a)>$
$P(A, B)=<P(a,b), P(a, \neg b), P(\neg a,b), P(\neg a,\neg b)>$

Bayesian Network



Bayesian network above represented by:

$$\begin{aligned} P(a, \neg b, c, \neg d) &= P(\neg d | a, \neg b, c) P(c | \neg b, a) P(\neg b | a) P(a) \\ &= P(\neg d | \neg b, c) P(c | \neg b) P(\neg b | a) P(a) \end{aligned}$$

Last time we discussed how to go left to right, when making the network

Today we look at right to left (inference)

Exact Inference

Our primary tool beyond this breakdown of P(a,b,c,d) is the sum rule:

$$P(b,c,d) = \sum_a P(a,b,c,d) = P(a,b,c,d) + P(\neg a,b,c,d)$$

We will also use the normalization trick for conditional probability (and not divide)

$$P(a,b) = \alpha P(a|b) \qquad P(a|b) = \alpha P(a,b)$$

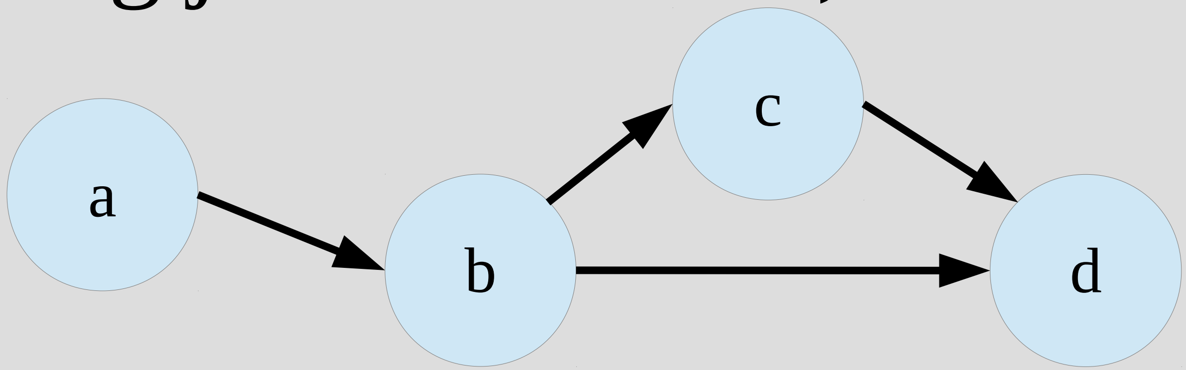
... or ...

$$\alpha(P(a|b) + P(\neg a|b)) = 1 \qquad P(a,b) + P(\neg a,b) = 1/\alpha$$

need to sum all non-given info

Exact Inference: Enumeration

Using just these facts, we can brute-force:



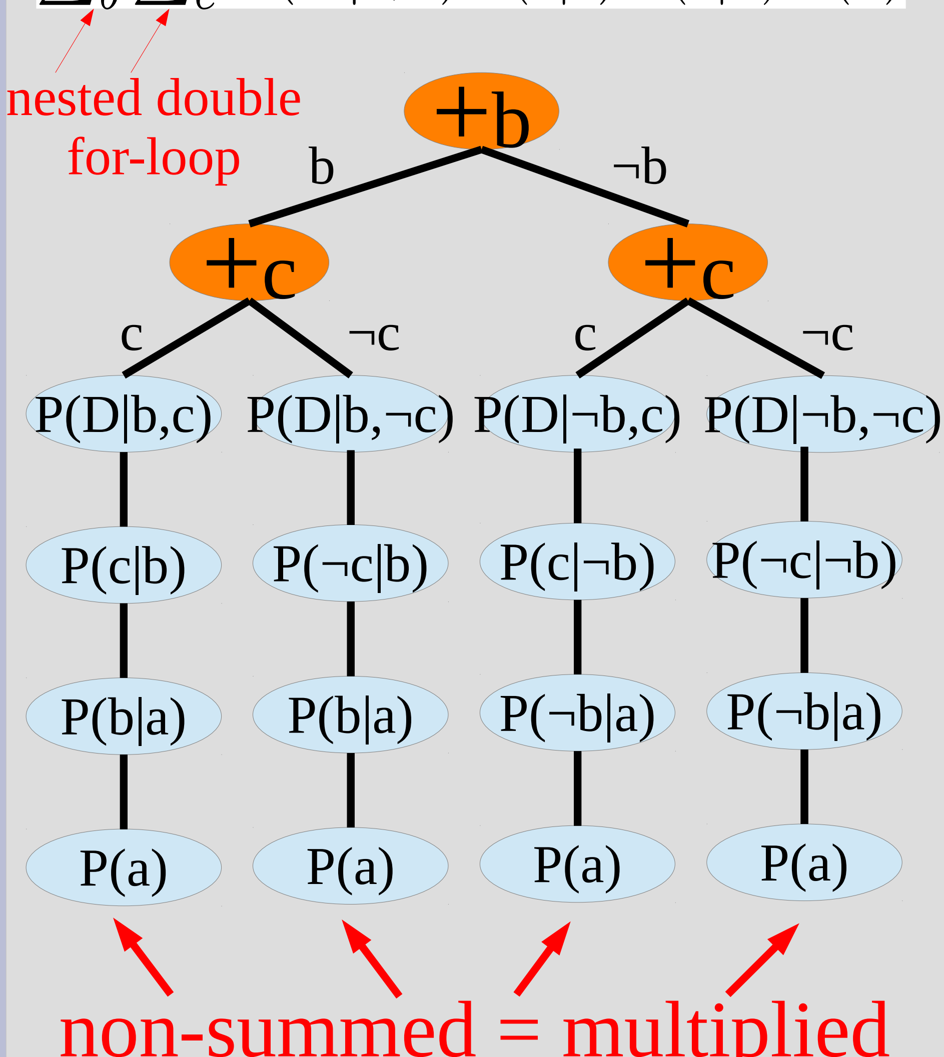
$$
\begin{aligned}
P(D|a) &= \sum_b \sum_c P(b, c, D|a) \\
&= \alpha \sum_b \sum_c P(a, b, c, D) \\
&= \alpha \sum_b \sum_c P(D|b, c) P(c|b) P(b|a) P(a) \\
&= \alpha P(a) \sum_b P(b|a) \sum_c P(c|b) P(D|b, c)
\end{aligned}
$$

Upper-case is both pos and neg (thus P(D|a) is array... here do formula twice) ... to find alpha

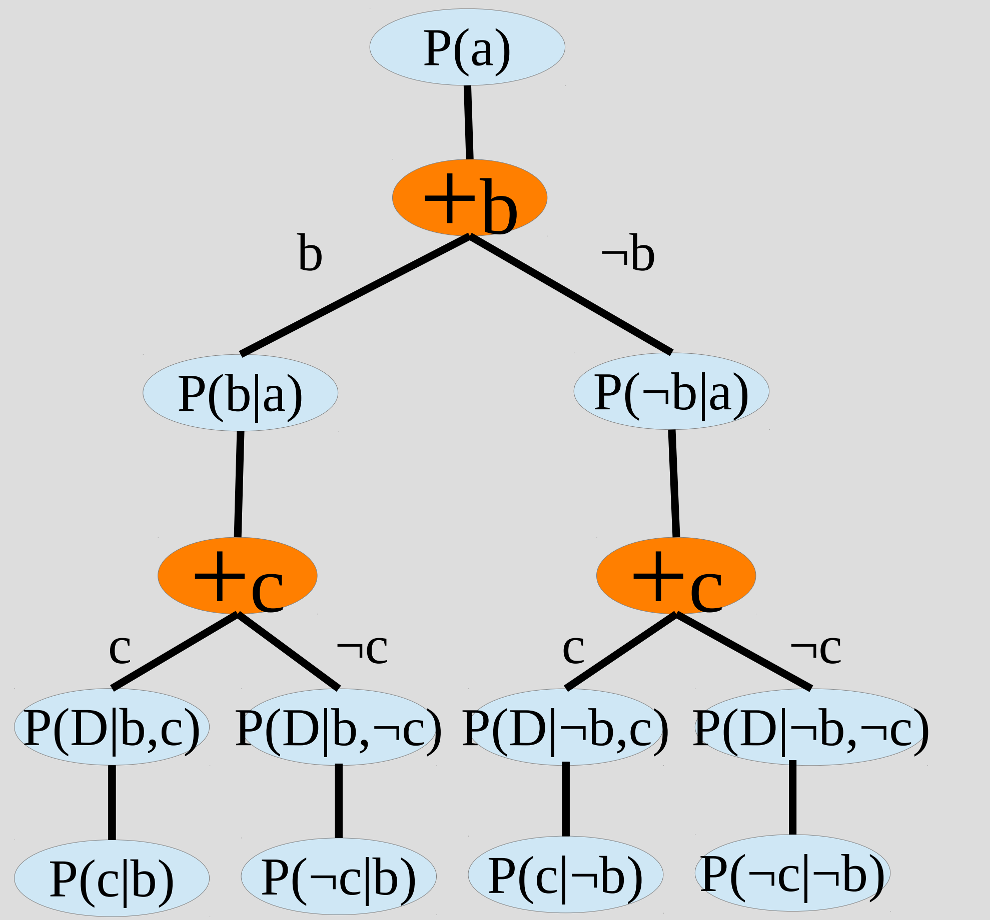
more efficient than previous

Exact Inference: Enumeration

$\sum_b \sum_c P(D|b,c)P(c|b)P(b|a)P(a)$

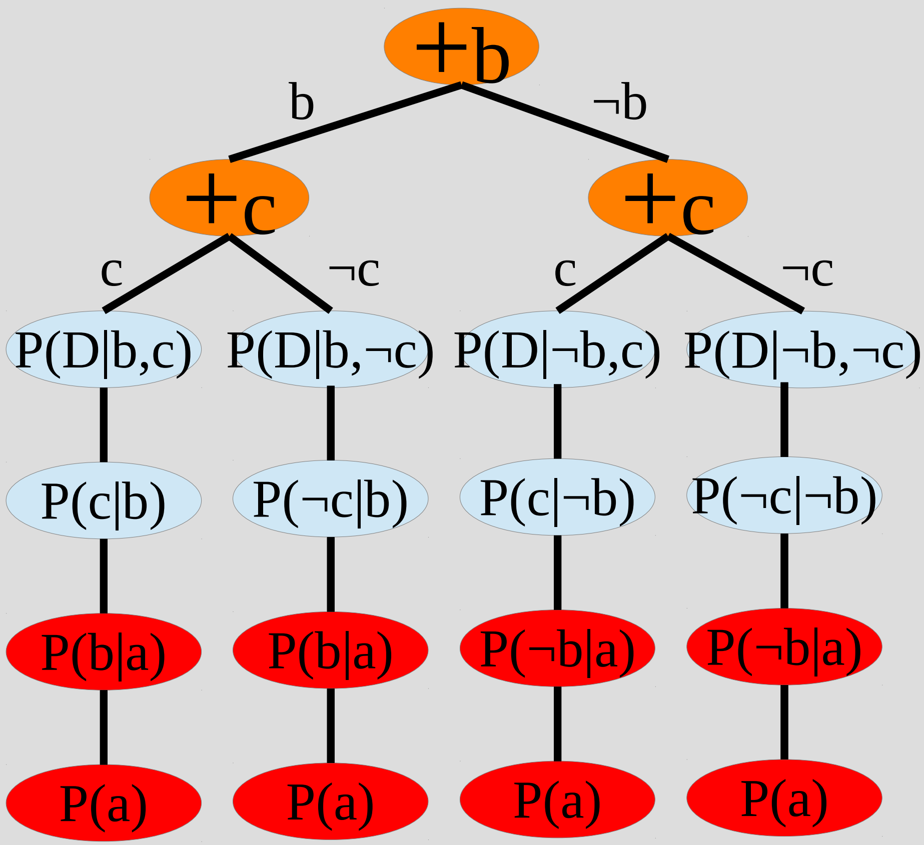


$P(a) \sum_b P(b|a) \sum_c P(D|b,c)P(c|b)$

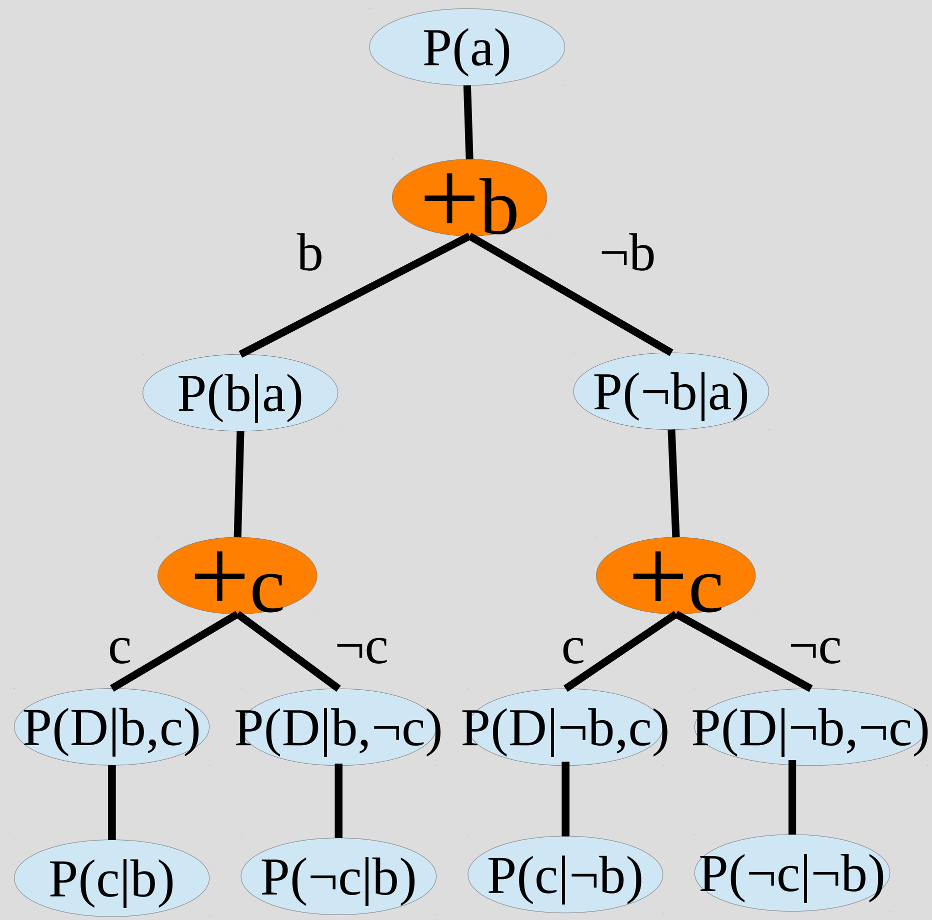


Exact Inference: Enumeration

$\sum_b \sum_c P(D|b,c)P(c|b)P(b|a)P(a)$



$P(a) \sum_b P(b|a) \sum_c P(D|b,c)P(c|b)$



Used in computation more than once (inefficient)

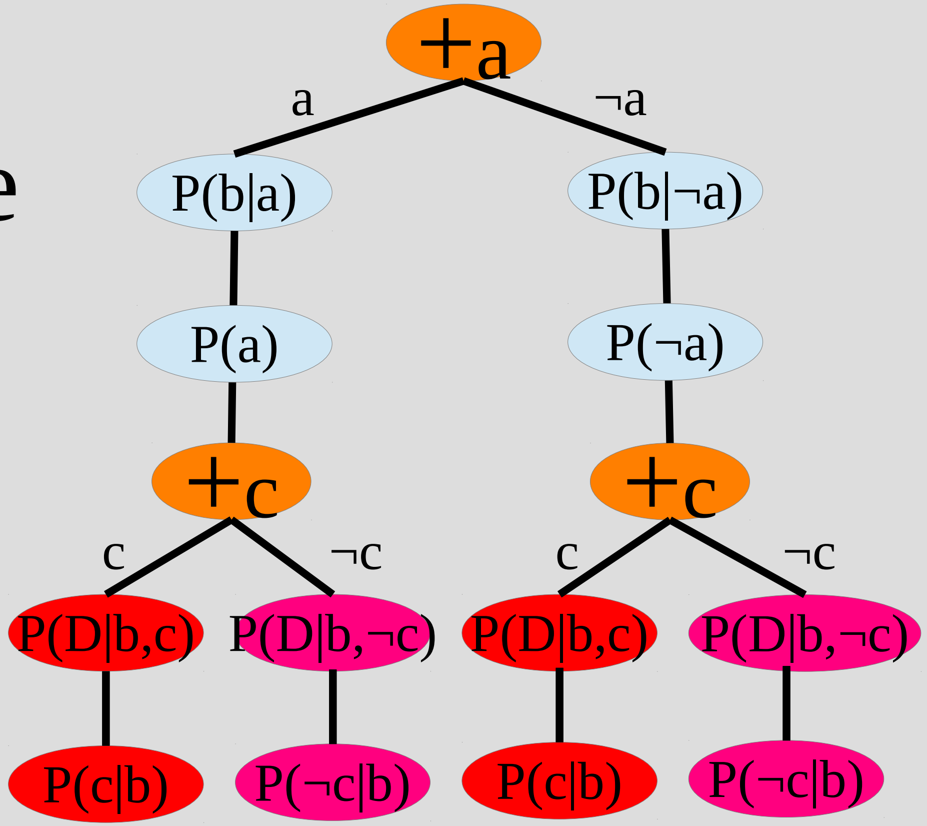
Exact Inference: Enumeration

We got lucky last time that we could eliminate all redundant calculations... not always so:

$$P(D|b) = \alpha \sum_a P(b|a)P(a) \sum_c P(D|b,c)P(c|b)$$

We can always eliminate all redundancy, but need another approach:

Dynamic programming



Dynamic Programming TL;DR

Two common ways to compute the Fibonacci numbers are (which is better?):

(1) Recursive (like prior slides: enumeration)

```
def fib(n):
    return fib(n-1) + fib(n-2)
```

(2) Array based (like upcoming slides)

```
a, b = 0, 1
while b < 50:
    a, b = b, a + b
```

Dynamic Programming TL;DR

Dynamic programming exploits the structure between parts of the problem

Rather than going top-down and having redundant computations along the way...

... dynamic programming goes bottom up and stores temporary results along the way

Exact Inference: Var. Elim.

Variable elimination is the dynamic programming version for Bayesian networks

This requires two new ideas:
(1) factors (denoted by “f”)
(2) “x” operator (called “pointwise product”)

Factors are the “stored info” that will represent the current product of probabilities

Exact Inference: Var. Elim.

Factors are basically partial truth-tables (or matrices) depending on “input” variables

The input variables: f(A,B) are what effects the factors (much like probability P(A,B))

When combing two factors with the “x” operator, the input variables are union-ed:

$$f_{\text{new}}(A, B, C) = f_1(A, B) \times f_2(A, C)$$

subscripts just help differentiate

Summing removes variables(like probabilities)

Exact Inference: Var. Elim.

How the "x" operation works is: multiply "matching" T/F values

or w/e type of values

$$f_{\text{new}}(A,B,C) = f_1(A,B) \times f_2(A,C)$$

For example (rand. numbers):

$$f_{\text{new}}(a,\neg b,c) = f_1(a,\neg b) \cdot f_2(a,c) = 0.34 \cdot 0.41 = 0.1394$$

$f_1(A,B)$

a	b	0.12
a	¬b	0.34
¬a	b	0.56
¬a	¬b	0.78

$f_2(A,C)$

a	c	0.41
a	¬c	0.52
¬a	c	0.63
¬a	¬c	0.74

$f_{\text{new}}(A,B,C)$

a	b	c	0.0492
a	b	¬c	0.0624
a	¬b	c	0.1394
a	¬b	¬c	0.1768
¬a	b	c	0.3528
¬a	b	¬c	0.4144
¬a	¬b	c	0.4914
¬a	¬b	¬c	0.5772

Exact Inference: Var. Elim.

Summation over the factors will work basically the same as probabilities:

or w/e type of values

$$f_{\text{new}}(A, B, C) = f_1(A, B) \times f_2(A, C)$$

You sum parts and remove it...

$$\sum_a f_1(A, B) = f_1(a, B) + f_1(\neg a, B) = f_{\text{new}}(B)$$

$f_1(A, B)$

a	b	0.12
a	¬b	0.34
¬a	b	0.56
¬a	¬b	0.78

$f_{\text{new}}(B)$

b	0.12 + 0.56
¬b	0.34 + 0.78

Exact Inference: Var. Elim.

Now we just represent the probabilities by factors and do "x" not normal multiplication

$$P(D|b) = \alpha \sum_a \underbrace{P(b|a)}_{f_1(A)} \underbrace{P(a)}_{f_2(A)} \sum_c \underbrace{P(D|b,c)}_{f_3(C,D)} \underbrace{P(c|b)}_{f_4(C)}$$

b is never negative, so not a variable

$$= \alpha \sum_a f_1(A) \times f_2(A) \times \sum_c f_3(C,D) \times f_4(C)$$

$$f_{3,4}(C,D) = f_3(C,D) \times f_4(C)$$

$$P(D|b) = \alpha \sum\nolimits_a f_1(A) \times f_2(A) \times \sum\nolimits_c f_{3,4}(C,D)$$

... then repeat "x" and sum (sum is normal sum over all T/F values (in this case))

Exact Inference: Var. Elim.

$$P(D|b) = \alpha \sum_a f_1(A) \times f_2(A) \times \sum_c f_{3,4}(C,D)$$

$$f_{3,4,c}(D) = f_{3,4}(c,D) + f_{3,4}(\neg c, D)$$

$$P(D|b) = \alpha \sum_a f_1(A) \times f_2(A) \times f_{3,4,c}(D)$$

$$f_{2,3,4,c}(A,D) = f_2(A) \times f_{3,4,c}(D)$$

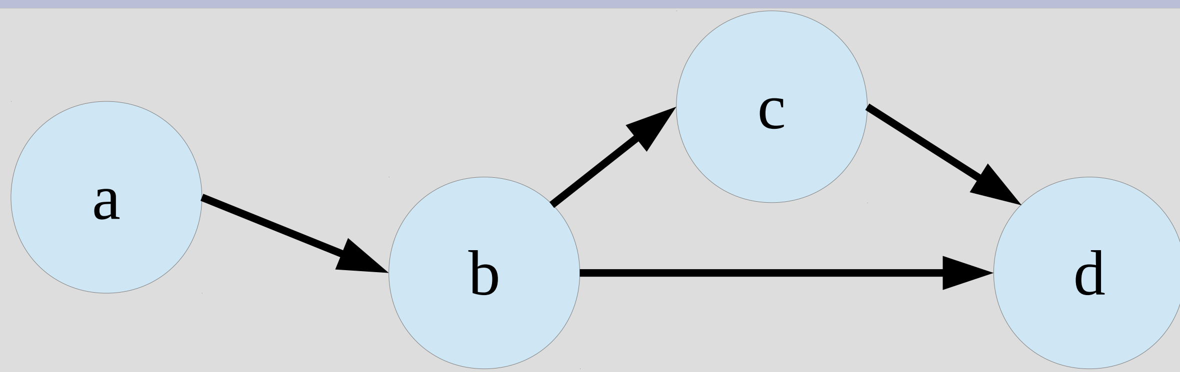
$$P(D|b) = \alpha \sum_a f_1(A) \times f_{2,3,4,c}(A,D)$$

$$P(D|b) = \alpha \sum_a f_{1,2,3,4,c}(A,D)$$

$$P(D|b) = \alpha f_{1,2,3,4,a,c}(D) =< P(d|b), P(\neg d|b) >$$

could also just call this f_5 or something

Exact Inference: Var. Elim.



P(a)	0.1

P(b|a)	0.2
P(b|¬a)	0.3

P(c|b)	0.4
P(c|¬b)	0.5

P(d|b,c)	0.25
P(d|b,¬c)	1.0
P(d|¬b,c)	0.15
P(d|¬b,¬c)	0.05

Using variable elimination, find:

$P(C|d, \neg a)$

$$P(C|\neg a, d) = \alpha \underbrace{P(\neg a)}_{f_1()} \sum_b \underbrace{P(C|b)}_{f_2(B,C)} \underbrace{P(d|b, C)}_{f_3(B,C)} \underbrace{P(b|\neg a)}_{f_4(B)}$$

$$f_{3,4}(b, c) = P(d|b, c)P(b|\neg a) = 0.25 \cdot 0.3 = 0.075$$
$$f_{3,4}(b, \neg c) = P(d|b, \neg c)P(b|\neg a) = 1 \cdot 0.3 = 0.3$$
$$f_{3,4}(\neg b, c) = P(d|\neg b, c)P(\neg b|\neg a) = 0.15 \cdot (1 - 0.3) = 0.105$$
$$f_{3,4}(\neg b, \neg c) = P(d|\neg b, \neg c)P(\neg b|\neg a) = 0.05 \cdot (1 - 0.3) = 0.035$$

$$P(C|\neg a, d) = \alpha f_1() \times \textstyle\sum_b f_2(B, C) \times f_{3,4}(B, C)$$

$$f_{2,3,4}(b, c) = P(c|b) * f(b, c) = 0.4 \cdot 0.075 = 0.03$$
$$f_{2,3,4}(b, \neg c) = P(\neg c|b) * f(b, \neg c) = (1 - 0.4) \cdot 0.3 = 0.18$$
$$f_{2,3,4}(\neg b, c) = P(c|\neg b) * f(\neg b, c) = 0.5 \cdot 0.105 = 0.0525$$
$$f_{2,3,4}(\neg b, \neg c) = P(\neg c|\neg b) * f(\neg b, \neg c) = (1 - 0.5) \cdot 0.035 = 0.0175$$

$$P(C|\neg a, d) = \alpha f_1() \times \textstyle\sum_b f_{2,3,4}(B, C)$$

$$P(C|\neg a, d) = \alpha f_1() \times \sum_b f_{2,3,4}(B, C)$$

$$f_{2,3,4,b}(c) = f_{2,3,4}(b, c) + f_{2,3,4}(\neg b, c) = 0.03 + 0.0525 = 0.0825$$

$$f_{2,3,4,b}(\neg c) = f_{2,3,4}(b, \neg c) + f_{2,3,4}(\neg b, \neg c) = 0.18 + 0.0175 = 0.1975$$

$$P(C|\neg a, d) = \alpha f_1() \times f_{2,3,4,b}(C)$$

$$f_{1,2,3,4,b}(c) = f_1() \cdot f_{2,3,4,b}(c) = P(\neg a) \cdot f_{2,3,4,b}(c) = 0.9 \cdot 0.0825 = 0.07425$$

$$f_{1,2,3,4,b}(\neg c) = f_1() \cdot f_{2,3,4,b}(\neg c) = P(\neg a) \cdot f_{2,3,4,b}(\neg c) = 0.9 \cdot 0.1975 = 0.17775$$

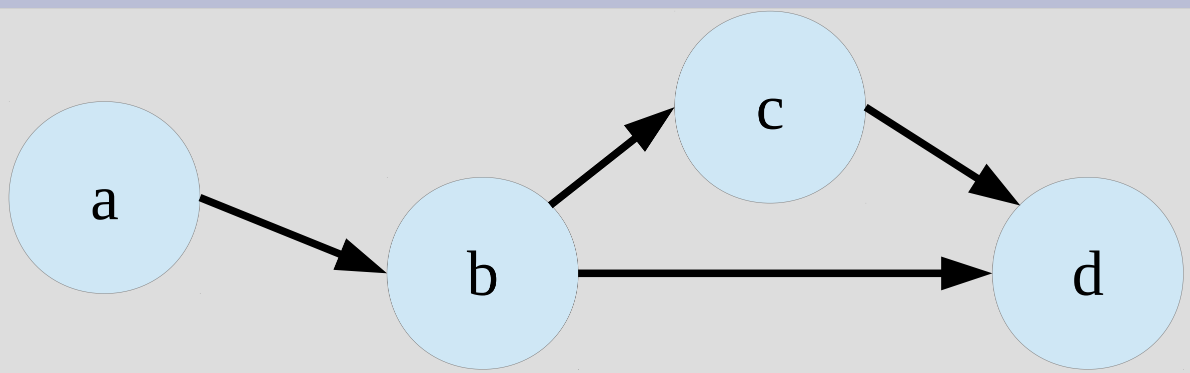
$$P(C|\neg a, d) = \alpha f_{1,2,3,4,b}(C)$$

normalize

$$P(c|\neg a, d) = 0.29464285714$$

$$\mathrm{P}(\neg c|\neg a, d) = 0.70535714286$$

Exact Inference: Side Note



If you try to find P(a|b) using either of these approaches:

$$
\begin{aligned}
P(a|b) &= \alpha \sum_c \sum_d P(d|b,c)P(c|b)P(\neg b|a)P(a) \\
&= \alpha P(b|a)P(a) \sum_c P(c|b) \sum_d P(d|\neg b, c) \\
&= \alpha P(b|a)P(a) \sum_c P(c|b) \cdot 1 \\
&= \alpha P(b|a)P(a)
\end{aligned}
$$

True for every non-ancestor of "b" or "a"

Bayes rule

Exact Inference: Var. Elim.

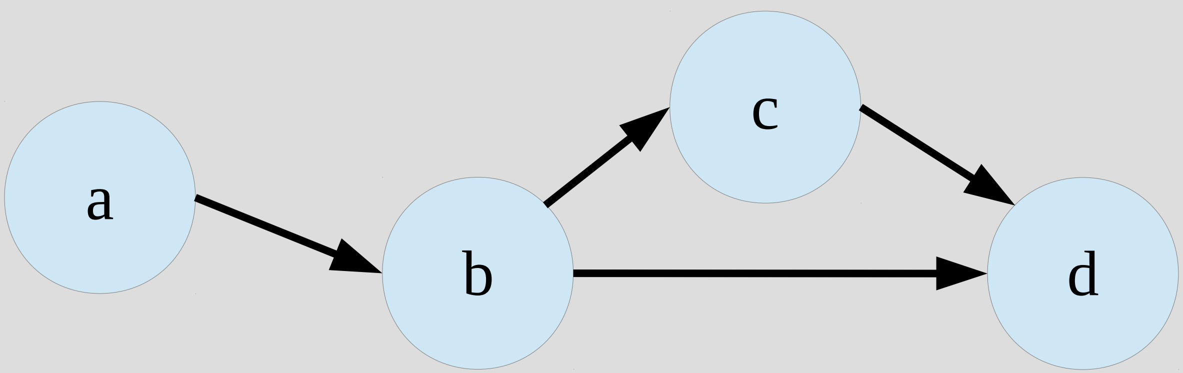
The order that you sum/combine factors can have a significant effect on runtime

However, there is no fast (i.e. worthwhile) way to compute the best ordering

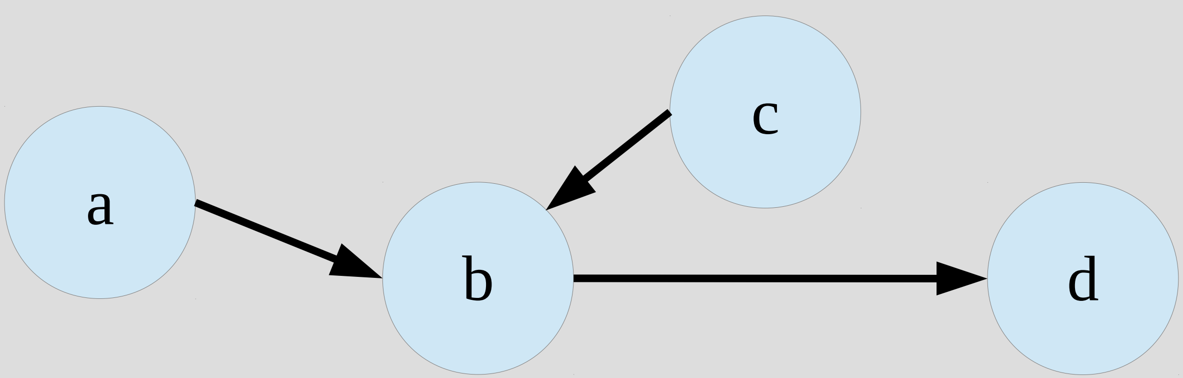
Instead, people quite often just use a greedy choice: combine/eliminate factors/variables to minimize resultant factor size

Efficiency

A <u>polytree</u> is a graph where there is at most one undirected path between nodes/variables



NOT polytree (two routes to d: b->c->d & b->d)



Yes, polytree (multiple roots are allowed)

Efficiency

Using the non-variable elimination way can result in exponential runtime

Using variable elimination:
On polytrees: Linear runtime
On non-polytrees: Exponential runtime :(

The details are a bit more nuanced, but basically exact inference is infeasible on non-polytrees (approximate methods for these)

Efficiency

You can do some preprocessing on graphs to cluster various parts:

The “b+c” node is much more complex (4 T/F value pairs, rather than a simple two T/F vals.)

Clustering can help when:

(1) Can be efficient to change into polytree

(2) Finding multiple probabilities

Efficiency

Not all nodes might be probabilistic

For example, if A is true then B is always true and if A false then B false (100% of the time)

Cases where nodes follow some formula (B=A), more efficient to not make a table

Two common formula are: <u>noisy-OR</u> and <u>noisy-max</u> (makes assumptions about parents)

Non-discrete

We have primarily stuck to true/false values for variables for simplicity sake

Variables could be any random variable (probability-value pair)

This includes continuous variables like normal/Gaussian distribution

Non-discrete

Sometimes you can discretize continuous variables (much like pixels or grids on map)

Otherwise you can use them directly and integrate instead of summing (yuck)

Things can get a bit complicated if the Bayesian network has both continuous and discrete variables

Non-discrete

Discrete parent of continuous:

-Simply do by cases

$$P(cont|disc) = \begin{cases} \frac{1}{\sqrt{2\pi}} e^{\frac{-(x-4)^2}{2}}, \text{ if } disc \\ \frac{1}{\sqrt{8\pi}} e^{\frac{-(x-4)^2}{8}}, \text{ if } \neg disc \end{cases}$$

Continuous to discrete:

-Have to correlate ranges with probabilities

$$P(disc|cont = x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$$

disc is true given cont has value x is:
percent under the normal(0,1) curve <= x