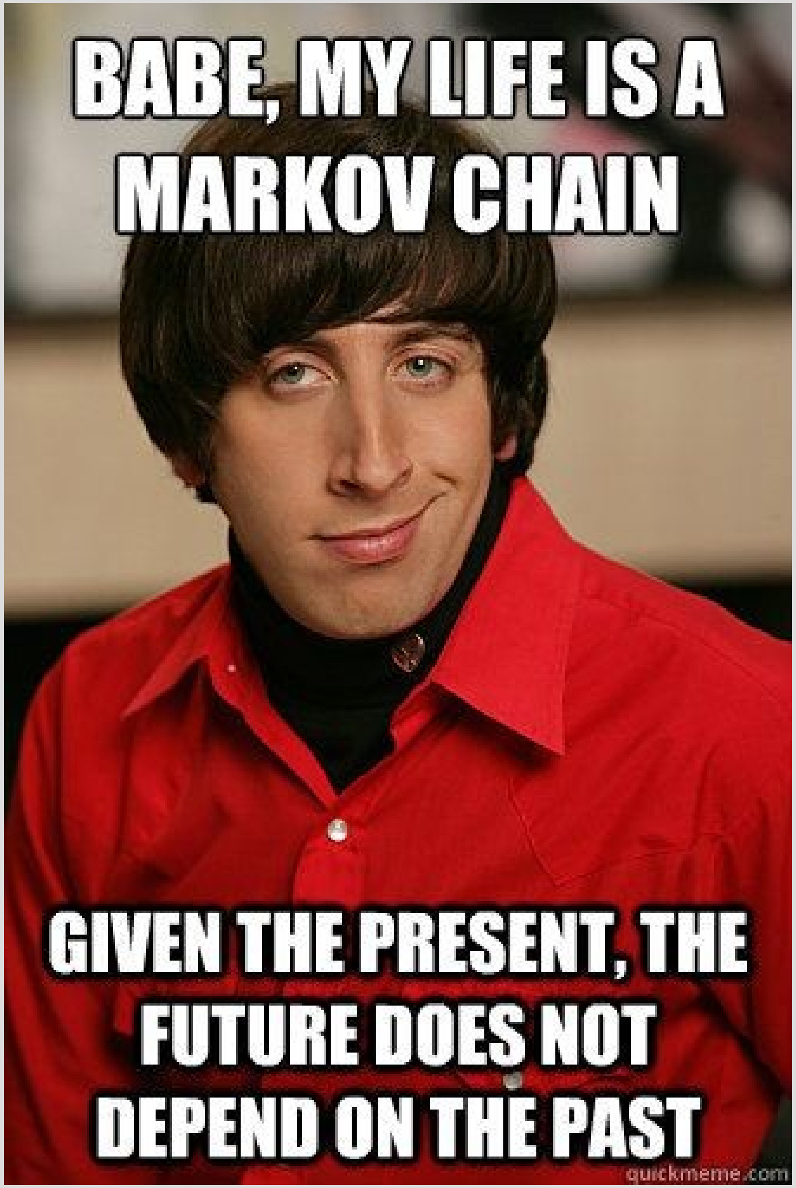
# Hidden Markov Models (Ch. 15)



# Why Gibbs works

To understand why Gibbs sampling works, we first need a bit more on Markov chains:

$$\pi_{t+1}(x') = \sum_x \pi_t(x) \cdot P(x \to x')$$

prob to get next state (e.g. [a,b,c])

prob in a state (e.g. [¬a,b,c])

prob change states (you just did this) (e.g. [¬a,b,c] → [a,b,c])

With the properties of irreducibility and aperiodicity, we will converge to a <u>stationary distribution</u> (i.e. stop changing)

$\pi_{t+1}(x) = \pi_t(x)$ (I will stop writing t's)

# Why Gibbs works

One way way to satisfy in-flow=out-flow is to simply say you must have equal flow between pairs of nodes

$$\pi(x)P(x \to x') = \pi(x')P(x' \to x)$$

From here it is enough to show that if you set:
π(x) = P(a,c,d|b), where x = {a,c,d}
P(x → x’) = P(x|MarkovBlanket(x))

... you will satisfy the stationary requirement

# Why Gibbs works

In our P(a,c,d|b) example:

$$\pi(a,c,d)P([a,c,d] \to [\neg a,c,d])$$
$$= P(a,c,d|b)P(\neg a|b,c,d)$$
$$= P(a|b,c,d)P(c,d|b)P(\neg a|b,c,d)$$
$$= P(a|b,c,d)P(\neg a,c,d|b)$$
$$= P([\neg a,c,d] \to [a,c,d])\pi(\neg a,c,d)$$

Thus we have our required property:

$$\pi(x)P(x \to x') = \pi(x')P(x' \to x)$$

# Why Gibbs works

In general:

$$\pi(x)P(x \rightarrow x')$$
$$= P(x|e)P(x_i'|\bar{x}_i, e)$$
$$= P(x_i, \bar{x}_i|e)P(x_i|\bar{x}_i, e)$$
$$= P(x_i|\bar{x}_i, e)P(\bar{x}_i|e)P(x_i'|\bar{x}_i, e)$$
$$= P(x_i|\bar{x}_i, e)P(x_i'|\bar{x}_i, e)$$
$$= P(x' \rightarrow x)\pi(x')$$

Note:

Technically, when finding P(x → x’) we have all variables as given, but we only use the Markov blanket as the other variables are conditionally independent

# Gibbs vs. Likelihood Weight

What are the differences (good and bad) between this method (Gibbs) and the one from last time (Likelihood Weighting)?

# Gibbs vs. Likelihood Weight

Good:
- Will not ever generate a 0 weight sample (as uses all evidence: P(c|a,b,d) not just parents in LW: P(c|b) )

Bad:
- Hard to tell when “converges” (no Law of Large Numbers to help bound error)
- Transition more unlikely if large blanket (as more probabilities multiplied = more variance)

# Zzzzz...

The rest of the chapter both:
- Gives real-ish world examples to use algs.
- Shows other ways of solving that (in general) not as good as using Bayesian networks

This is kinda boring so I will skip all except the last part on “Fuzzy logic”

# Fuzzy Logic

So far we have been saying things like:
A=true ... or ... OverAte=true

Fuzzy logic moves away from true/false and instead makes these continuous variables, so:
OverAte=0.4 is possible

This is <u>not</u> a 40% chance you overate,
it is more like your stomach is 40% full
(a known fact, not a thing of chance)

# Fuzzy Logic

You can define basic logic operators in Fuzzy logic as well:

(A or B) = max(A,B)

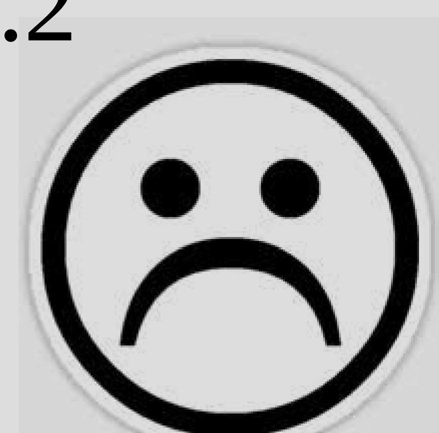
(A and B) = min(A,B)

(¬A) = 1-(A)

... So if OverAte=0.4 and Desert=0.2

(OverAte or Desert) = 0.4

However, (Desert or ¬Desert)=0.8



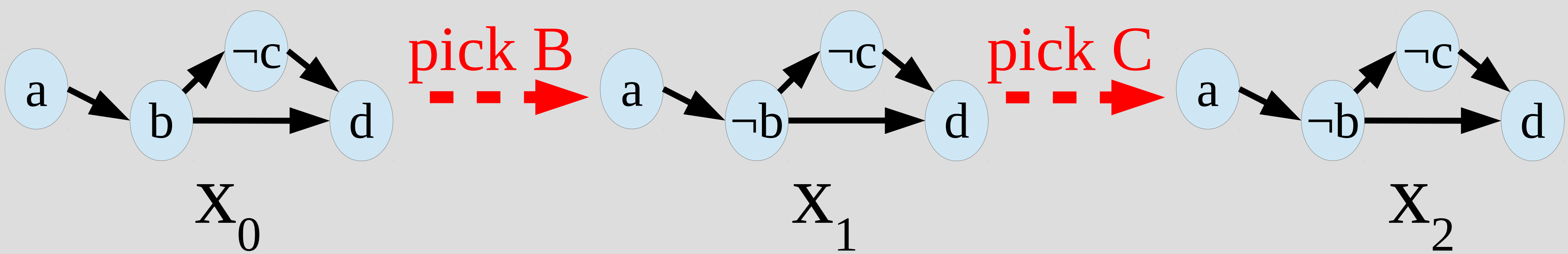

NEXT TOPIC

# Markov... chains?

Recap, Markov property:

$$P(x_{n+1}|x_n, x_{n-1}, x_{n-2} \cdots x_0) = P(x_{n+1}|x_n)$$

(Next state only depends on current state)




For Gibbs sampling, we made a lot of samples using the Markov property (since this is 1-dimension, it looks like a “chain”)

# Markov... chains?

For the next bit, we will still have a “Markov” and uncertainty (i.e. probabilities)

However, we will add partial-observability (some things we cannot see)

These are often called Hidden Markov Models (not “chains” because they won’t be 1D... w/e)

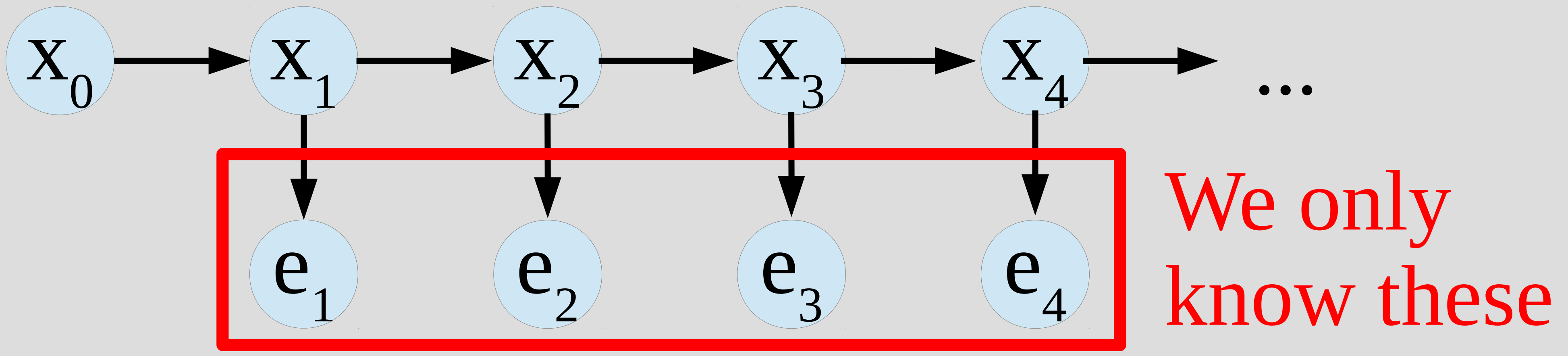
# Hidden Markov Models

For Hidden Markov Models (HMMs) often:

(1) E = the evidence

(2) X = the hidden/not observable part

We assume the hidden information is what causes the evidence (otherwise quite easy)



# Hidden Markov Models

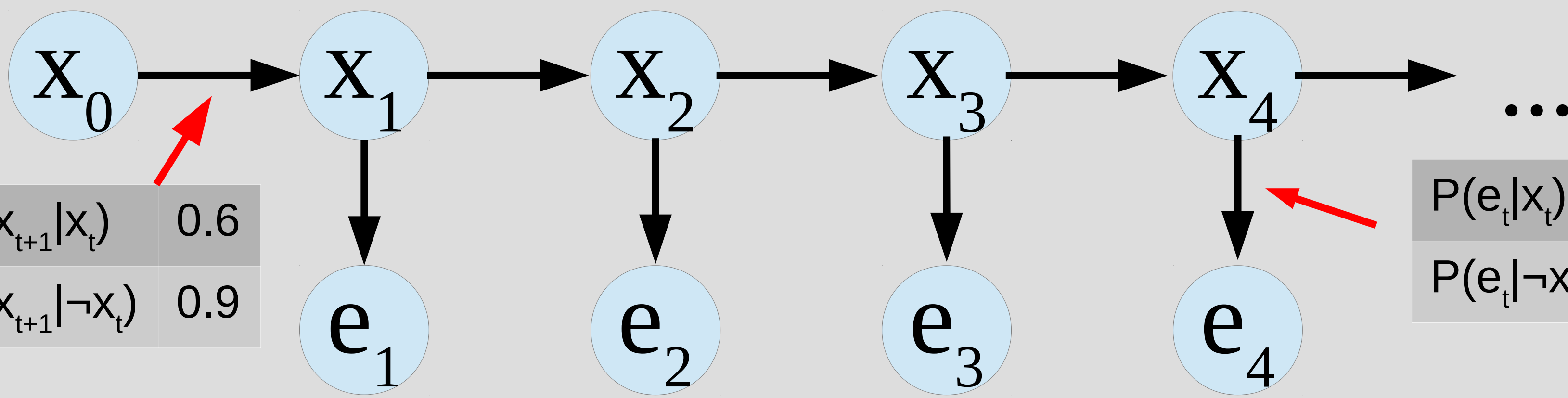
Our example will be: sleep deprivation


So variable $X_t$ will be if a person got enough sleep on day t

This person is not you, but you see them every day, and you can tell if their eyes are bloodshot (this is $E_t$)

# Hidden Markov Models

If you squint a bit, this is actually a Bayesian network as well (though can go on for a while)




| | |
|---|---|
| $P(x_{t+1}\|x_t)$ | 0.6 |
| $P(x_{t+1}\|\neg x_t)$ | 0.9 |

| | |
|---|---|
| $P(e_t\|x_t)$ | 0.3 |
| $P(e_t\|\neg x_t)$ | 0.8 |

For simplicity's sake, we will assume the probabilities of going to the right (next state) and down (seeing evidence) are the same for all subscripts (typically "time")

# Hidden Markov Models

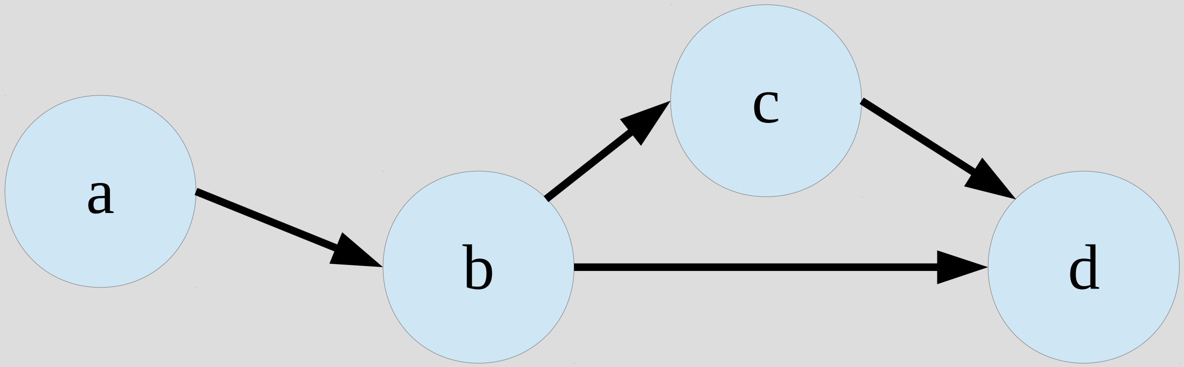
As we will be dealing with quite a few variables, we will introduce some notation:

$E_{1:t} = E_1, E_2, E_3, \ldots. E_t$ (similarly for $X_{0:t}$)
So $P(E_{1:t}) = P(E_1, E_2, E_3, .. E_t)$, which is normal definition of commas like P(a,b)

We will assume we only know $E_{1:t}$ (and $X_0$) and want to figure out $X_k$ for various k

# Hidden Markov Models

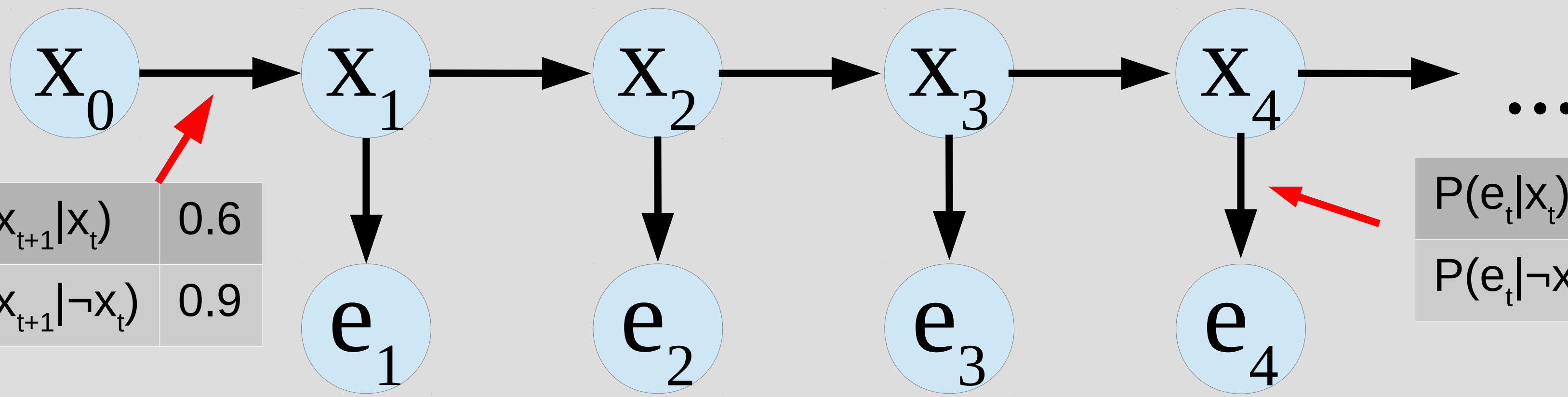
Quick Bayesian network recap:




$$
\begin{aligned}
P(a,\neg b,c,\neg d) &= P(\neg d|a,\neg b,c)P(c|\neg b,a)P(\neg b|a)P(a)\\
&= P(\neg d|\neg b,c)P(c|\neg b)P(\neg b|a)P(a)\\
&= \boxed{\prod_{x\in Network} P(x|Parents(x))}
\end{aligned}
$$

Used fact tons in our sampling...

# Hidden Markov Models




| | |
|---|---|
| $P(x_{t+1}\|x_t)$ | 0.6 |
| $P(x_{t+1}\|\neg x_t)$ | 0.9 |

| | |
|---|---|
| $P(e_t\|x_t)$ | 0.3 |
| $P(e_t\|\neg x_t)$ | 0.8 |

So in this Bayesian network (bigger):

$$P(X_{0:t}, E_{1:t}) = P(X_0) \prod_{i=1}^{t} P(X_i|X_{i-1})P(E_i|X_i)$$

Most Likely Explanation

Typically, use above to compute four things:

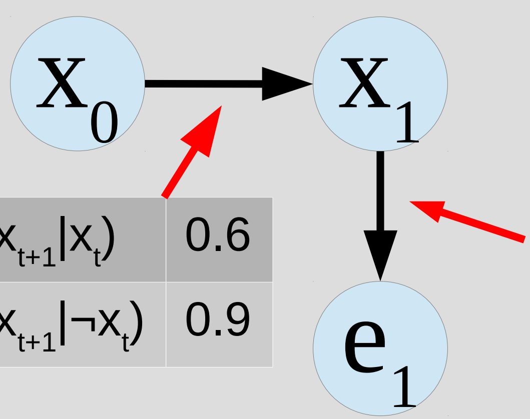
| Filtering | Prediction | Smoothing | MLE |
|---|---|---|---|
| $P(x_t\|e_{1:t})$ | $P(x_{t+k}\|e_{1:t})$ | $P(x_k\|e_{1:t}), k < t$ | $P(x_{1:t}\|e_{1:t})$ |

# Filtering in HMMs

All four of these are actually quite similar, and you can probably already find them

The only difficulty is the size of the Bayesian network, so let's start small to get intuition:

| | |
|---|---|
| $P(x_0)$ | 0.5 |




| | |
|---|---|
| $P(x_{t+1}\|x_t)$ | 0.6 |
| $P(x_{t+1}\|\neg x_t)$ | 0.9 |

How can you find $P(x_1|\neg e_1)$?

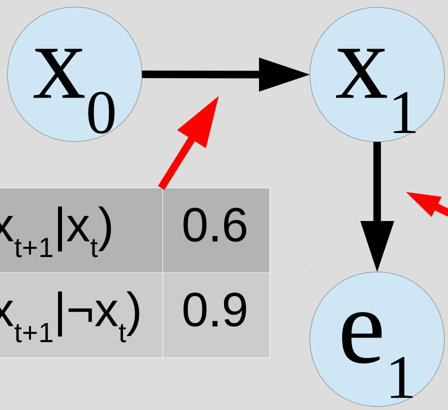
(this is a simple Bays-net)

| | |
|---|---|
| $P(e_t\|x_t)$ | 0.3 |
| $P(e_t\|\neg x_t)$ | 0.8 |

$$P(X_{0:t}, E_{1:t}) = P(X_0) \prod_{i=1}^{t} P(X_i|X_{i-1}) P(E_i|X_i)$$

# Filtering in HMMs

| | |
|---|---|
| $P(x_0)$ | 0.5 |




| | |
|---|---|
| $P(x_{t+1}\|x_t)$ | 0.6 |
| $P(x_{t+1}\|\neg x_t)$ | 0.9 |

| | |
|---|---|
| $P(e_t\|x_t)$ | 0.3 |
| $P(e_t\|\neg x_t)$ | 0.8 |

$$
\begin{aligned}
P(x_1|\neg e_1) &= \alpha P(x_1, \neg e_1) \\
&= \alpha \sum_{x_0} P(x_0, x_1, \neg e_1) \\
&= \alpha \sum_{x_0} P(x_0) P(x_1|x_0) P(\neg e_1|x_1) \\
&= \alpha P(\neg e_1|x_1) \sum_{x_0} P(x_0) P(x_1|x_0) \\
&= \alpha (1 - 0.3) \cdot (0.5 \cdot 0.6 + 0.5 \cdot 0.9) \\
&\approx 0.525\alpha
\end{aligned}
$$

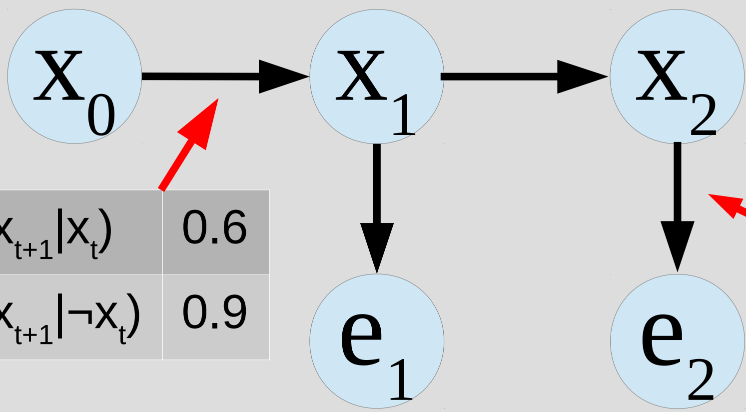
Similarly, $P(\neg x_1|\neg e_1) \approx 0.05\alpha$

So normalized gives: $P(x_1|\neg e_1) \approx 0.913$

91% chance I slept last night, given today I didn't have bloodshot eyes

# Filtering in HMMs

| $P(x_0)$ | 0.5 |
|---|---|




| | |
|---|---|
| $P(x_{t+1} \mid x_t)$ | 0.6 |
| $P(x_{t+1} \mid \neg x_t)$ | 0.9 |

| | |
|---|---|
| $P(e_t \mid x_t)$ | 0.3 |
| $P(e_t \mid \neg x_t)$ | 0.8 |

Find: $P(x_2 \mid \neg e_1, \neg e_2)$

# Filtering in HM

$$P(x_2|\neg e_1, \neg e_2) = \alpha P(x_2, \neg e_1, \neg e_2)$$

Double sum?!?!
Double... for loop..?

$$= \alpha \sum_{x_0} \sum_{x_1} P(x_0, x_1, x_2 \neg e_1, \neg e_2)$$

$$= \alpha \sum_{x_0} \sum_{x_1} P(x_0) P(x_1|x_0) P(\neg e_1|x_1) P(x_2|x_1) P(\neg e_2|x_2)$$

$$= \alpha \sum_{x_1} \sum_{x_0} P(\neg e_2|x_2) P(x_2|x_1) P(\neg e_1|x_1) P(x_0) P(x_1|x_0)$$

Just computed this!
It is $P(x_1|e_1)$

$$= \alpha P(\neg e_2|x_2) \sum_{x_1} P(x_2|x_1) \boxed{P(\neg e_1|x_1) \sum_{x_0} P(x_0) P(x_1|x_0)}$$

$$= \alpha P(\neg e_2|x_2) \big(P(x_2|x_1) \cdot 0.913 + P(x_2|\neg x_1) \cdot (1 - 0.913)\big)$$

$$= \alpha (1 - 0.3)(0.6 \cdot 0.913 + 0.9 \cdot (1 - 0.913))$$

$$\approx 0.438\alpha$$

... after normalizing you should get: ≈0.854

# Filtering in HMMs

Double sum?!?!
Double... for loop..?

$$
\begin{aligned}
P(x_2|\neg e_1, \neg e_2) &= \alpha P(x_2, \neg e_1, \neg e_2) \\
&= \alpha \sum_{x_0} \sum_{x_1} P(x_0, x_1, x_2 \neg e_1, \neg e_2) \\
&= \alpha \sum_{x_0} \sum_{x_1} P(x_0) P(x_1|x_0) P(\neg e_1|x_1) P(x_2|x_1) P(\neg e_2|x_2) \\
&= \alpha \sum_{x_1} \sum_{x_0} P(\neg e_2|x_2) P(x_2|x_1) P(\neg e_1|x_1) P(x_0) P(x_1|x_0) \\
&= \alpha P(\neg e_2|x_2) \sum_{x_1} P(x_2|x_1) \boxed{P(\neg e_1|x_1) \sum_{x_0} P(x_0) P(x_1|x_0)} \\
&= \alpha P(\neg e_2|\boxed{x_2}) \left( P(\boxed{x_2}|x_1) \cdot 0.913 + P(\boxed{x_2}|\neg x_1) \cdot (1 - 0.913) \right) \\
&= \alpha (1 - 0.3)(0.6 \cdot 0.913 + 0.9 \cdot (1 - 0.913)) \\
&\approx 0.438\alpha
\end{aligned}
$$

Just computed this!
It is $P(x_1|e_1)$

change to $\neg x_2$

$P(\neg x_2|\neg e_1,\neg e_2)$
$\approx 0.075\alpha$

... after normalizing you should get: ≈0.854

# Filtering in HMMs

In general:

$$\begin{aligned}
P(x_t|e_{1:t}) &= P(x_t|e_t, e_{1:t-1}) \\
&= \alpha P(x_t, e_t, e_{1:t-1}) \\
&= \alpha P(x_t, e_t|e_{1:t-1}) \qquad \text{(Note: different } \alpha\text{)} \\
&= \alpha P(e_t|x_t, e_{1:t-1}) P(x_t|e_{1:t-1}) \\
&= \alpha P(e_t|x_t, e_{1:t-1}) \sum_{x_{t-1}} \Big( P(x_t, x_{t-1}|e_{1:t-1}) \Big) \\
&= \alpha P(e_t|x_t, e_{1:t-1}) \sum_{x_{t-1}} \Big( P(x_t|x_{t-1}, e_{1:t-1}) P(x_{t-1}|e_{1:t-1}) \Big) \\
&= \alpha P(e_t|x_t) \sum_{x_{t-1}} \Big( P(x_t|x_{t-1}) P(x_{t-1}|e_{1:t-1}) \Big)
\end{aligned}$$

# Filtering in HMMs

In general:

$$
\begin{aligned}
\boxed{P(x_t|e_{1:t})} &= P(x_t|e_t, e_{1:t-1}) \\
&= \alpha P(x_t, e_t, e_{1:t-1}) \\
&= \alpha P(x_t, e_t|e_{1:t-1}) \qquad \text{(Note: different } \alpha\text{)} \\
&= \alpha P(e_t|x_t, e_{1:t-1}) P(x_t|e_{1:t-1}) \\
&= \alpha P(e_t|x_t, e_{1:t-1}) \sum_{x_{t-1}} \Big( P(x_t, x_{t-1}|e_{1:t-1}) \Big) \\
&= \alpha P(e_t|x_t, e_{1:t-1}) \sum_{x_{t-1}} \Big( P(x_t|x_{t-1}, e_{1:t-1}) P(x_{t-1}|e_{1:t-1}) \Big) \\
&= \alpha P(e_t|x_t) \sum_{x_{t-1}} \Big( P(x_t|x_{t-1}) \boxed{P(x_{t-1}|e_{1:t-1})} \Big)
\end{aligned}
$$

... same, but different ‘t’

# Filtering in HMMs

In general:

$$
\begin{aligned}
\boxed{f(t)} &= P(x_t|e_t, e_{1:t-1}) \\
&= \alpha P(x_t, e_t, e_{1:t-1}) \\
&= \alpha P(x_t, e_t|e_{1:t-1}) \qquad \text{(Note: different } \alpha\text{)} \\
&= \alpha P(e_t|x_t, e_{1:t-1}) P(x_t|e_{1:t-1}) \\
&= \alpha P(e_t|x_t, e_{1:t-1}) \sum_{x_{t-1}} \Big( P(x_t, x_{t-1}|e_{1:t-1}) \Big) \\
&= \alpha P(e_t|x_t, e_{1:t-1}) \sum_{x_{t-1}} \Big( P(x_t|x_{t-1}, e_{1:t-1}) P(x_{t-1}|e_{1:t-1}) \Big) \\
&= \alpha P(e_t|x_t) \sum_{x_{t-1}} \Big( P(x_t|x_{t-1}) \boxed{f(t-1)} \Big)
\end{aligned}
$$

Actually, this is just a recursive function

# Filtering in HMMs

So we can compute f(t) = $P(x_t|e_{1:t})$:

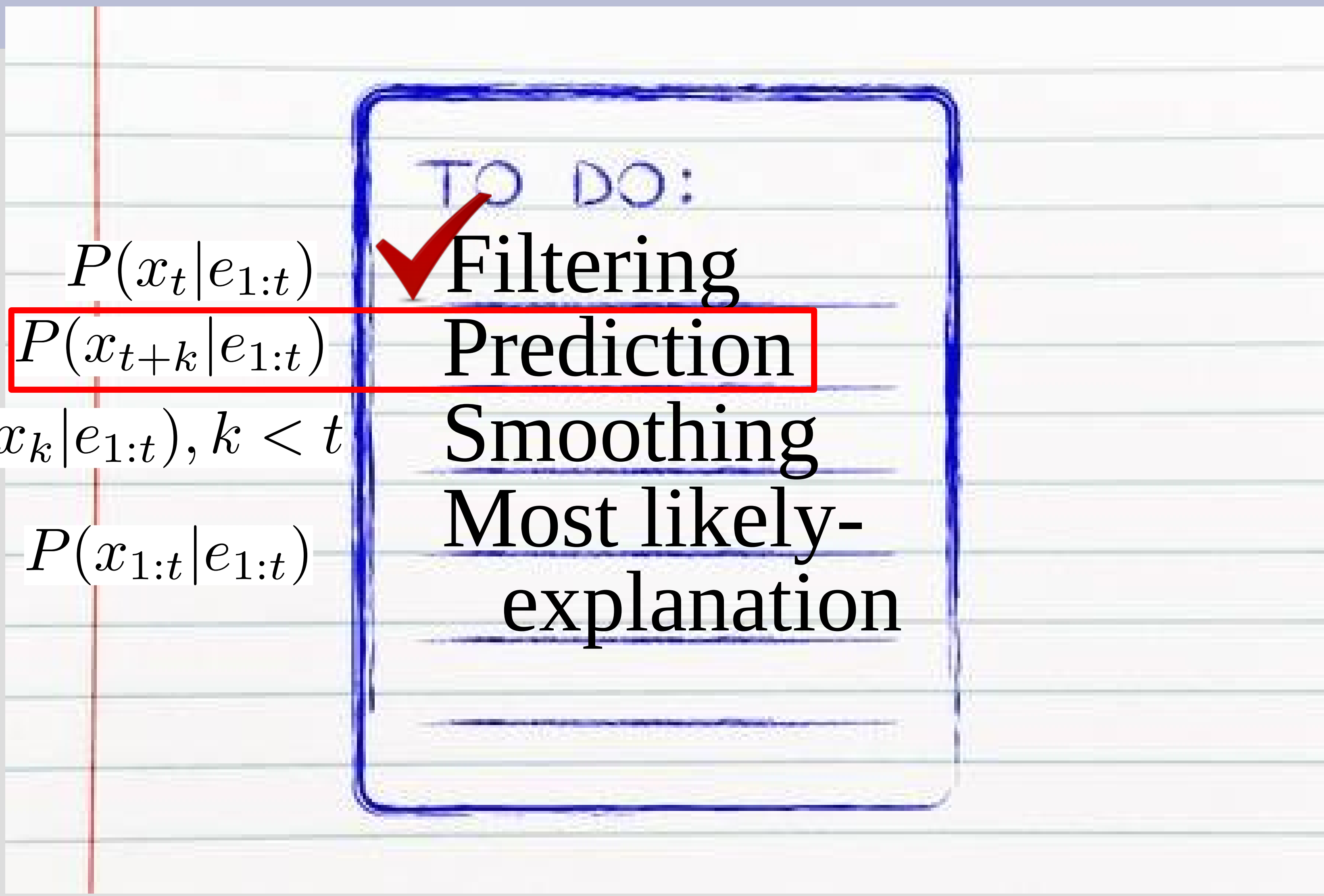
$$f(t) = \alpha P(e_t|x_t) \sum_{x_{t-1}} \Big( P(x_t|x_{t-1}) f(t-1) \Big)$$

actually an array, as you need both T/F for sum(or 1-)

Of course, we don't *actually* want to do this recursively... rather with dynamic programing

Start with f(0) = $P(x_0)$, then use this to find f(1)... and so on (can either store in array, or just have a single variable... like Fibonacci)

# Prediction in HMMs



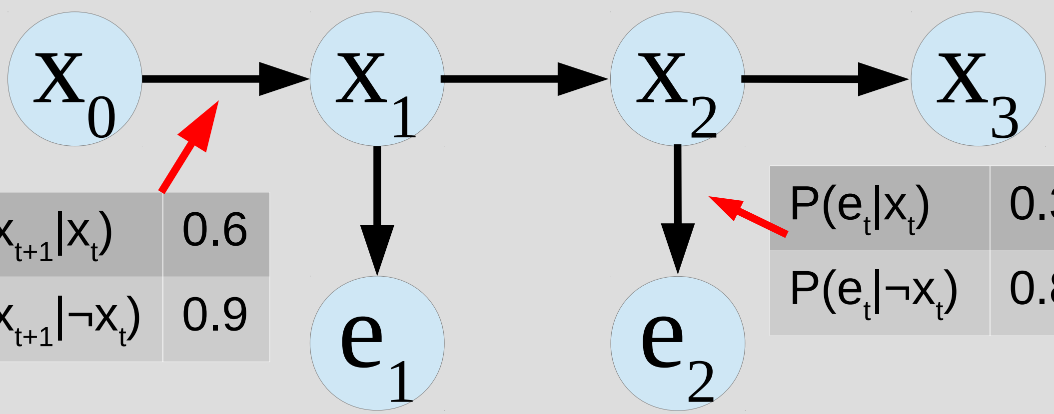

- $P(x_t|e_{1:t})$ — Filtering
- $P(x_{t+k}|e_{1:t})$ — Prediction
- $P(x_k|e_{1:t}), k < t$ — Smoothing
- $P(x_{1:t}|e_{1:t})$ — Most likely-explanation

How would you find “prediction”?

# Prediction in HMMs

Probably best to go back to the example:
What is chance I sleep on day 3 given, you saw me without bloodshot eyes on day 1&2?

| | |
|---|---|
| $P(x_0)$ | 0.5 |




| | |
|---|---|
| $P(x_{t+1} \mid x_t)$ | 0.6 |
| $P(x_{t+1} \mid \neg x_t)$ | 0.9 |

| | |
|---|---|
| $P(e_t \mid x_t)$ | 0.3 |
| $P(e_t \mid \neg x_t)$ | 0.8 |

$$P(x_3 \mid \neg e_1, \neg e_2) = ???$$

# Prediction in HMMs

$$
\begin{aligned}
P(x_3|\neg e_1, \neg e_2) &= \alpha P(x_3, \neg e_1, \neg e_2) \\
&= \alpha \sum_{x_0} \sum_{x_1} \sum_{x_2} P(x_0, x_1, x_2, x_3, \neg e_1, \neg e_2) \\
&= \alpha \sum_{x_0} \sum_{x_1} \sum_{x_2} P(x_0) P(x_1|x_0) P(\neg e_1|x_1) P(x_2|x_1) P(e_2|x_2) P(x_3|x_2) \\
&= \alpha \sum_{x_2} P(x_3|x_2) \boxed{P(e_2|x_2) \sum_{x_1} P(x_2|x_1) P(\neg e_1|x_1) \sum_{x_0} P(x_0) P(x_1|x_0)} \\
&= \alpha \sum_{x_2} P(x_3|x_2) \boxed{P(x_2|\neg e_1, \neg e_2)} \\
&= \alpha (0.6 \cdot 0.854 + 0.9 \cdot (1 - 0.854)) \\
&\approx 0.644\alpha
\end{aligned}
$$

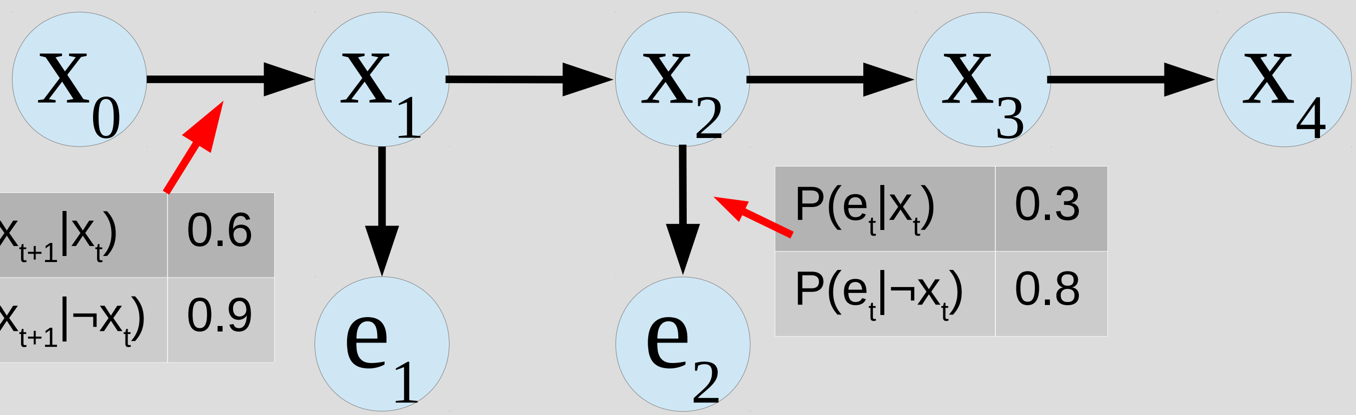
whew...

Turns out that $P(\neg x_3|\neg e_1, \neg e_2) \approx 0.356\alpha$, so $\alpha=1$

# Prediction in HMMs

Day 4?

| | |
|---|---|
| $P(x_0)$ | 0.5 |




| | |
|---|---|
| $P(x_{t+1} \mid x_t)$ | 0.6 |
| $P(x_{t+1} \mid \neg x_t)$ | 0.9 |

| | |
|---|---|
| $P(e_t \mid x_t)$ | 0.3 |
| $P(e_t \mid \neg x_t)$ | 0.8 |

$$P(x_4 \mid \neg e_1, \neg e_2) = ???$$

# Prediction in HMMs

$$
\begin{aligned}
P(x_4|\neg e_1, \neg e_2) &= \alpha P(x_4, \neg e_1, \neg e_2) \\
&= \alpha \sum_{x_0} \sum_{x_1} \sum_{x_2} \sum_{x_3} P(x_0, x_1, x_2, x_3, x_4, \neg e_1, \neg e_2) \\
&= \alpha \sum_{x_0} \sum_{x_1} \sum_{x_2} \sum_{x_3} P(x_0) P(x_1|x_0) P(\neg e_1|x_1) P(x_2|x_1) P(\neg e_2|x_2) P(x_3|x_2) P(x_4|x_3) \\
&= \alpha \sum_{x_3} P(x_4|x_3) \boxed{\sum_{x_2} P(x_3|x_2) P(\neg e_2|x_2) \sum_{x_1} P(x_2|x_1) P(\neg e_1|x_1) \sum_{x_0} P(x_0) P(x_1|x_0)} \\
&= \alpha \sum_{x_3} P(x_4|x_3) \boxed{P(x_3|\neg e_1, \neg e_2)} \\
&= \alpha(0.6 \cdot 0.644 + 0.9 \cdot (1 - 0.644)) \\
&\approx 0.707\alpha
\end{aligned}
$$

...think I see a pattern here

Turns out that $P(\neg x_4|\neg e_1,\neg e_2) \approx 0.293\alpha$, so $\alpha=1$

($\alpha$ always 1 now, as can move into red box)

# Prediction in HMMs

TO DO:

- $P(x_t|e_{1:t})$ ✓ Filtering
- $P(x_{t+k}|e_{1:t})$ ✓ Prediction
- $P(x_k|e_{1:t}), k < t$ Smoothing
- $P(x_{1:t}|e_{1:t})$ Most likely-explanation

We'll save the other two for next time...