GENERAL VECTOR SPACES AND SUBSPACES [4.1]

General vector spaces

➤ So far we have seen special spaces of vectors of n dimensions – denoted by $\mathbb{R}^n$.

➤ It is possible to define more general vector spaces

A vector space V over $\mathbb{R}$ is a nonempty set with two operations:

- Addition denoted by $'+'$. For two vectors x and y, $x + y$ is a member of V
- Multiplication by a scalar For $\alpha \in \mathbb{R}$ and $x \in V$, αx is a member of V.

➤ In addition for V to be a vector space the following 8 axioms must be satisfied [note: order is different in text]

1. Addition is commutative $u + v = v + u$
2. Addition is associative $u + (v + w) = (u + v) + w$
3. $\exists$ zero vector denoted by 0 such that $\forall u$, $0 + u = u$
4. Any u has an opposite $-u$ such that $u + (-u) = 0$
5. $1u = u$ for any u
6. $(\alpha\beta)u = \alpha(\beta u)$
7. $(\alpha + \beta)u = \alpha u + \beta u$
8. $\alpha(u + v) = \alpha u + \alpha v$

✍ Show that the zero vector in Axiom 3 is unique, and the vector $-u$, ('negative of u'), in Axiom 4 is unique for each u in V.

- For each u in V and scalar α we have

$$0u = 0 \qquad \alpha 0 = 0\ ; \qquad -u = (-1)u\ .$$

Example: Let V be the set of all arrows (directed line segments) in three-dimensional space, with two arrows regarded as equal if they have the same length and point in the same direction. Define addition by the parallelogram rule, and for each v in V, define cv to be the arrow whose length is c times the length of v, pointing in the same direction as v if $c > 0$ and otherwise pointing in the opposite direction.

Note: The definition of V is geometric, using concepts of length and direction. No xyz-coordinate system is involved. An arrow of zero length is a single point and represents the zero vector. The negative of v is $(-1)v$.

- All axioms are verified

More examples

- Set of vectors in $\mathbb{R}^4$ with second component equal to zero.
- Set of all poynomials of degree ≤ 3
- Set of all $m \times n$ matrices
- Set of all $n \times n$ upper triangular matrices

Subspaces

➤ A subset $\boldsymbol{H}$ of vectors of $\boldsymbol{V}$ is a subspace if it is a vector space by itself. Formal definition:

➤ A subset $\boldsymbol{H}$ of vectors of $\boldsymbol{V}$ is a subspace if

1. $\boldsymbol{H}$ is closed for the addition, which means:

$$\boldsymbol{x}+\boldsymbol{y} \in \boldsymbol{H} \quad \text{for any} \quad \boldsymbol{x} \in \boldsymbol{H}, \boldsymbol{y} \in \boldsymbol{H}$$

2. $\boldsymbol{H}$ is closed for the scalar multiplication, which means:

$$\boldsymbol{\alpha x} \in \boldsymbol{H} \quad \text{for any} \quad \boldsymbol{\alpha} \in \mathbb{R}, \boldsymbol{x} \in \boldsymbol{H}$$

➤ Note: If $\boldsymbol{H}$ is a subspace then (1) $\mathbf{0}$ belongs to $\boldsymbol{H}$ and (2) For any $\boldsymbol{x} \in \boldsymbol{H}$, the vector $-\boldsymbol{x}$ belongs to $\boldsymbol{H}$

➤ Every vector space is a subspace (of itself and possibly of other larger spaces).

➤ The set consisting of only the zero vector of V is a subspace of V, called the zero subspace. Notation: $\{0\}$.

Example: Polynomials of the form

$$p(t) = \alpha_2 t^2 + \alpha_3 t^3$$

form a subspace of the space of polynomials of degree ≤ 3

✍ Other examples: Examples 3 and 5 (sec. 4.1) from *text*

Example: Triangular matrices

✍ Example 8 (sec. 4.1) in *text* is important

✍ Show that the set $\boldsymbol{H}$ of all vectors in $\mathbb{R}^3$ of the form $\{\boldsymbol{a}+\boldsymbol{b}, \boldsymbol{a}-\boldsymbol{b}, \boldsymbol{b}\}$ is a subspace of $\mathbb{R}^3$. [Hint: see example 11 from Sec. 4.1 of *text*]

➤ Recall: the term linear combination refers to a sum of scalar multiples of vectors, and $\mathbf{span}\{\boldsymbol{v}_1, ..., \boldsymbol{v}_p\}$ denotes the set of all vectors that can be written as linear combinations of $\boldsymbol{v}_1, \cdots, \boldsymbol{v}_p$.

A subspace spanned by a set

Theorem: *If $v_1, ..., v_p$ are in a vector space V, then*

$$\text{span}\{v_1, ..., v_p\}$$

is a subspace of V.

➤ $\text{span}\{v_1, ..., v_p\}$ is the subspace spanned (or generated) by $\{v_1, ..., v_p\}$.

➤ Given any subspace H of V, a spanning (or generating) set for H is a set $\{v_1, ..., v_p\}$ in H such that $H = \text{span}\{v_1, ...v_p\}$.

✍ Prove above theorem for $p = 2$, i.e., given v_1 and v_2 in a vector space V, then $H = \text{span}\{v_1, v_2\}$ is a subspace of V. [Hint: show that H is closed for '$+$' and for scalar multiplication]

NULL SPACES AND COLUMN SPACES [4.2]

Null space of a matrix

Definition: The null space of an $m \times n$ matrix A, written as Nul(A), is the set of all solutions of the homogeneous equation $Ax = 0$. In set notation,

$$\text{Nul}(A) = \{x : x \in \mathbb{R}^n \quad \text{and} \quad Ax = 0\}.$$

Theorem: The null space of an $m \times n$ matrix A is a subspace of $\mathbb{R}^n$

➤ Equivalently, the set of all solutions to a system $Ax = 0$ of m homogeneous linear equations in n unknowns is a subspace of $\mathbb{R}^n$

Proof: Nul(A) is by definition a subset of $\mathbb{R}^n$. Must show: Nul(A) closed under $+$ and multipl. by scalars.

➤ Take u and v any two vectors in Nul(A). Then $Au = 0$ and $Av = 0$.

➤ Need to show that $\boldsymbol{u}+\boldsymbol{v}$ is in Nul($\boldsymbol{A}$), i.e., that $\boldsymbol{A}(\boldsymbol{u}+\boldsymbol{v})=\boldsymbol{0}$. Using a property of matrix multiplication, compute

$$\boldsymbol{A}(\boldsymbol{u}+\boldsymbol{v})=\boldsymbol{A}\boldsymbol{u}+\boldsymbol{A}\boldsymbol{v}=\boldsymbol{0}+\boldsymbol{0}=\boldsymbol{0}$$

➤ Thus $\boldsymbol{u}+\boldsymbol{v} \in$ Nul($\boldsymbol{A}$), and Nul($\boldsymbol{A}$) is closed under vector addition.

➤ Finally, if $\boldsymbol{\alpha}$ is any scalar, then $\boldsymbol{A}(\boldsymbol{\alpha}\boldsymbol{u})=\boldsymbol{\alpha}(\boldsymbol{A}\boldsymbol{u})=\boldsymbol{\alpha}(\boldsymbol{0})=\boldsymbol{0}$ which shows that $\boldsymbol{\alpha}\boldsymbol{u}$ is in Nul($\boldsymbol{A}$).

➤ Thus Nul($\boldsymbol{A}$) is a subspace of $\mathbb{R}^{\boldsymbol{n}}$. ■

✍ See Example 1 in Sect. 4.2 of *text* [determining if a given vector belongs to Nul($\boldsymbol{A}$)

✍ See Example 2 in Sect. 4.2 of *text* [determining a subspace by casting as a null space]

➤ Next we will see how to determine Nul($\boldsymbol{A}$). See Example 3 of Sec. 4.2 of *text*. Details next.

➤ There is no obvious relation between vectors in Nul($\boldsymbol{A}$) and the entries in $\boldsymbol{A}$.

➤ We say that Nul($\boldsymbol{A}$) is defined implicitly, because it is defined by a condition that must be checked.

➤ No explicit list or description of the elements in Nul($\boldsymbol{A}$), so..

➤ ... we need to solve the equation $\boldsymbol{Ax} = \boldsymbol{0}$ to produce an explicit description of Nul($\boldsymbol{A}$).

Example: Find the null space of the matrix

$$\boldsymbol{A} = \begin{bmatrix} -3 & 6 & -1 & 1 & -7 \\ 1 & -2 & 2 & 3 & -1 \\ 2 & -4 & 5 & 8 & -4 \end{bmatrix}$$

➤ We will find a spanning set for Nul($\boldsymbol{A}$).

Solution: first step is to find the general solution of $Ax = 0$ in terms of free variables. We know how to do this.

- Get reduced echelon form of augmented matrix $[A\ 0]$:

$$\begin{bmatrix} 1 & -2 & 0 & -1 & 3 & 0 \\ 0 & 0 & 1 & 2 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{aligned} x_1 - 2x_2 \quad - x_4 + 3x_5 &= 0 \\ x_3 + 2x_4 - 2x_5 &= 0 \\ 0 &= 0 \end{aligned}$$

- x_2, x_4, x_5 are free variables, x_1, x_3 basic variables.

- For any selection of the free variables, can find a vector in Nul(A) by computing x_1, x_3 in terms of these variables:

$$\begin{aligned} x_1 &= 2x_2 + x_4 - 3x_5 \\ x_3 &= -2x_4 + 2x_5 \end{aligned}$$

➤ OK - but how can we write these using spanning vectors (i.e. as linear combinations of specific vectors?)

➤ Solution - write x as:

$$\begin{vmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{vmatrix} = \begin{vmatrix} 2x_2 & +x_4 & -3x_5 \\ x_2 & & \\ & -2x_4 & +2x_5 \\ & x_4 & \\ & & x_5 \end{vmatrix} = x_2 \underbrace{\begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}}_{u} + x_4 \underbrace{\begin{bmatrix} 1 \\ 0 \\ -2 \\ 1 \\ 0 \end{bmatrix}}_{v} + x_5 \underbrace{\begin{bmatrix} -3 \\ 0 \\ 2 \\ 0 \\ 1 \end{bmatrix}}_{w}$$

➤ General solution is of the form $x_2 u + x_4 v + x_5 w$.

➤ Every linear combination of u, v, and w is an element of Nul(A). Thus $\{u, v, w\}$ is a spanning set for Nul(A), i.e.,

$$\text{Nul}(A) = \text{span}\{u, v, w\}$$

✍ Obtain the vector $\boldsymbol{x}$ of Nul($\boldsymbol{A}$)corresponding to the choice: $\boldsymbol{x_2 = 1, x_4 = -2, x_5 = -1}$. Verify that indeed it is in the null space, i.e., that $\boldsymbol{Ax = 0}$

✍ For same example, find a vector in Nul($\boldsymbol{A}$)whose last two components are zero and whose first component is 1. How many such vectors are there (zero, one, or inifintely many?)

Notes:

➤ 1. The spanning set produced by the method in the example is guaranteed to be linearly independent

✍ Show this (proof by contradiction)

➤ 2. When Nul($\boldsymbol{A}$)contains nonzero vectors, the number of vectors in the spanning set for Nul($\boldsymbol{A}$) equals the number of free variables in the equation $\boldsymbol{Ax = 0}$.

Column Space of a matrix

Definition: The column space of an $m \times n$ matrix A, written as Col(A) (or $C(A)$), is the set of all linear combinations of the columns of A. If $A = [a_1 \cdots a_n]$, then

$$\text{Col}(A) = \mathbf{span}\{a_1, ..., a_n\}$$

Theorem: The column space of an $m \times n$ matrix A is a subspace of $\mathbb{R}^m$.

➤ A vector in Col(A) can be written as Ax for some x [Recall that Ax stands for a linear combination of the columns of A].

That is:

$$\text{Col}(A) = \{b : b = Ax \quad \text{for some } x \text{ in } \quad \mathbb{R}^n\}$$

➤ The notation $\boldsymbol{Ax}$ for vectors in Col($\boldsymbol{A}$) also shows that Col($\boldsymbol{A}$) is the range of the linear transformation $\boldsymbol{x} \rightarrow \boldsymbol{Ax}$.

➤ The column space of an $\boldsymbol{m} \times \boldsymbol{n}$ matrix $\boldsymbol{A}$ is all of $\mathbb{R}^m$ if and only if the equation $\boldsymbol{Ax} = \boldsymbol{b}$ has a solution for each $\boldsymbol{b}$ in $\mathbb{R}^m$

✍ Let

$$\boldsymbol{A} = \begin{bmatrix} 2 & 4 & -2 & 1 \\ -2 & -5 & 7 & 3 \\ 3 & 7 & -8 & 6 \end{bmatrix}, \quad \boldsymbol{u} = \begin{bmatrix} 3 \\ -2 \\ -1 \\ 0 \end{bmatrix}, \quad \boldsymbol{v} = \begin{bmatrix} 3 \\ -1 \\ 3 \end{bmatrix}$$

a. Determine if $\boldsymbol{u}$ is in Nul($\boldsymbol{A}$). Could $\boldsymbol{u}$ be in Col($\boldsymbol{A}$)?

b. Determine if $\boldsymbol{v}$ is in Col($\boldsymbol{A}$). Could $\boldsymbol{v}$ be in Nul($\boldsymbol{A}$) ?

➤ General remarks and hints:

1. Col(A) is a subspace of $\mathbb{R}^m$ [$m = 3$ in above example]

2. Nul(A) is a subspace of $\mathbb{R}^n$ [$n = 4$ in above example]

3. To verify that a given vector x belongs to Nul(A) all you need to do is check if $Ax = 0$

4. To verify if $b \in$ Col(A) all you need to do is check if the linear system $Ax = b$ has a solution.