Graph Laplacians

D L Boley
University of Minnesota

Graph Analysis based on Laplacian

- Many properties of a graph can be obtained or estimated from properties of the so-called Laplacian matrix.
  - average hitting times, commute times.
  - distances or affinities between nodes.
  - effective resistances for passive electrical network.
  - relative importance of nodes on web: pagerank.
  - bottlenecks in computer communication networks, road networks.
  - minimal graph cuts.
  - behavior of consensus dynamics.

- Much existing theory is for undirected graphs
- Some can be extended to directed graphs.
- Much of this material is from [Boley et al., 2010].

Undirected vs Directed graphs

Undirected Graph

- social networks:
  friends and contact lists
- passive electrical networks
- recommender systems:
  e.g. bipartite graph:
  users $\leftrightarrow$ movies.
- the internet, computer communication networks.

Directed Graph

- the WWW: random walk on relaxed graph yields pagerank.
- road network with one-way streets.
- wireless device network with mix of high and low-powered devices.

Outline

- Graphs Matrices & Laplacians
- Average Hitting and Commute Times
- Embedding in Euclidean Space
- Electric Resistances
- Spectral Graph Partitioning: Cuts
- Cheeger-like bounds.
- Conclusions

Basics: Graphs and Matrices

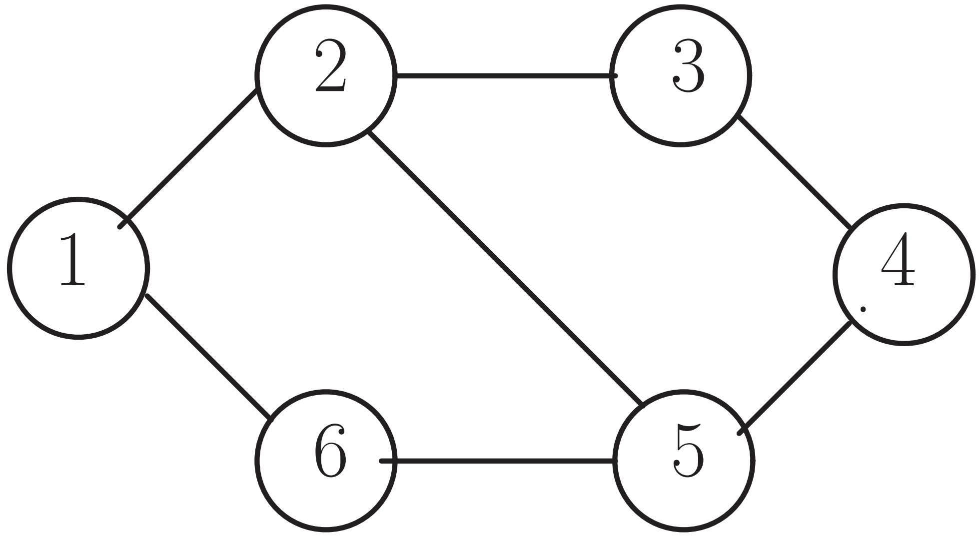
- Graph represented by
  - Adjacency Matrix A s.t. $a_{ij} \neq 0$ when $\exists$ an edge $i \to j$.
  - Markov chain transition matrix P s.t. p_{ij} = probability of transition from node i to node j.
  - Undirected graph $\iff$ symmetric adjacency matrix $\iff$ reversible Markov chain.
  - Assume no absorbing states $\iff$ strongly connected.
- Related Quantities
  - $\mathbf{d} = A \cdot \mathbf{1}$ vector of node (out) degrees,
  - $D = \mathrm{diag}(\mathbf{d})$ = diagonal matrix of degrees,
  - $\boldsymbol{\pi}$ = vector of stationary probabilities, s.t. $\boldsymbol{\pi}^{\mathrm{T}} P = \boldsymbol{\pi}^{\mathrm{T}}$,
  - Π = diagonal matrix of stationary probabilities,
  - $Z = (I - P + \mathbf{1}\boldsymbol{\pi}^{\mathrm{T}})^{-1}$ = Fundamental Matrix [Grinstead & Snell, 2006].

Alternative Laplacians

Laplacians lead to many graph properties (many for undirected graphs)

- $L^{\mathrm{a}} = D - A = D(I - P)$ "combinatorial," based on node degrees.
  - Matrix Tree Theorem $\rightarrow$ number of spanning 'trees' anchored at each node (DiGraphs too) [Brualdi & Ryser, 1991; Chebotarev & Shamis, 2006]
  - smallest graph cut relative to number of nodes in each half [Shi & Malik, 2000; Spielman & Teng, 1996; von Luxburg, 2007].
- $L = \Pi(I - P)$ "Random Walk" $= L^{\mathrm{a}} \cdot$ vol*2 if undirected.
  - pseudo-inverse leads to average commute times/resistances [Doyle & Snell, 1984; Chandra et al., 1989; Klein & Randic, 1993; Boley et al., 2011].
  - pseudo-inverse leads to metric embedding in $\mathbb{R}^n$ [Gower & Legendre, 1986; Fouss et al., 2007].
- $L^{\mathrm{p}} = I - P = I - D^{-1}A = D^{-1}L^{\mathrm{a}}$ "normalized"
  - smallest graph cut relative to number of edges in each half [von Luxburg, 2007].
  - Consensus dynamics over nodes of a graph: $\dot{\mathbf{x}} = -L\mathbf{x}$ (DiGraphs too). [Olfati-Saber et al., 2004, 2006], [Bamieh et al., 2008], [Young et al., 2010, 2011].
- $\mathcal{L} = D^{1/2} L^{\mathrm{p}} D^{-1/2} = D^{-1/2} L^{\mathrm{a}} D^{-1/2} =$ symmetrized normalized Laplacian.
  - shares same eigenvalues as $L^{\mathrm{p}} = I - P$.

Example – Undirected Graph



$$A = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 \end{pmatrix} \qquad \mathbf{d} = \begin{pmatrix} 2 \\ 3 \\ 2 \\ 2 \\ 3 \\ 2 \end{pmatrix} \qquad \boldsymbol{\pi} = \frac{1}{14} \cdot \begin{pmatrix} 2 \\ 3 \\ 2 \\ 2 \\ 3 \\ 2 \end{pmatrix}$$

Laplacians

- $L^{\mathrm{a}} = \begin{pmatrix} 2 & -1 & 0 & 0 & 0 & -1 \\ -1 & 3 & -1 & 0 & -1 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & -1 & 0 & -1 & 3 & -1 \\ -1 & 0 & 0 & 0 & -1 & 2 \end{pmatrix} = 14 * L$
- Number of spanning 'trees': $\det(L^{\mathrm{a}}_{[2:6],[2:6]}) = 15$.
- Eigenvalues are 0, 1, 2, 3, 3, 5.
- Eigenvector corresp. to 1 (Fiedler vector): $(1, 0, -1, -1, 0, 1)/2$. Used in Spectral Graph Partitioning.
- Volume = number of edges = $^1\!/_2 \mathrm{trace}(L^{\mathrm{a}}) = 7$.

Outline

- Graphs Matrices & Laplacians
- Average Hitting and Commute Times
- Embedding in Euclidean Space
- Electric Resistances
- Spectral Graph Partitioning: Cuts
- Cheeger-like bounds.
- Conclusions

Number of Visits

- Partition $P = \left[\begin{array}{c|c} P_{11} & \mathbf{p}_{12} \\ \hline \mathbf{p}_{21}^T & p_{nn} \end{array}\right]$.
- If last row replaced with $[\mathbf{0}^T, 1]$, then $[P_{11}^k]_{ij}$ is the probability of being in node j starting in node i at the $k-th$ step, before reaching n.
- $[I + P_{11} + P_{11}^2 + \cdots]_{ij} = [(I - P_{11})^{-1}]_{ij} \stackrel{\text{def}}{=} N(i,j,n)$
$= \#$ visits to j starting from i before reaching n.
- $(I - P_{11})^{-1} = [\Pi_{1,\ldots,n-1}^{-1} \underbrace{\Pi_{1,\ldots,n-1}(I - P_{11})}_{L_{11}}]^{-1} = L_{11}^{-1}\Pi_{1,\ldots,n-1}$.
- Since $L \cdot \mathbf{1} = \mathbf{0}$, $\mathbf{1}^T L = \mathbf{0}^T$,
lemma (next page) yields $N(i,j,n) = (m_{ij} + m_{nn} - m_{in} - m_{nj})\pi_j$.
- Choice of destination node n is arbitrary, so
$N(i,j,k) = (m_{ij} + m_{kk} - m_{ik} - m_{kj})\pi_j$ for all i,j,k.

Lemma 1 – Inverse of Submatrix

Let $L = \begin{pmatrix} L_{11} & \mathbf{l}_{12} \\ \mathbf{l}_{21}^{\mathrm{T}} & l_{nn} \end{pmatrix}$ be an $n \times n$ irreducible matrix s.t. $\mathsf{nullity}(L) = 1$.
Let $M = L^{+}$ be the pseudo-inverse of L partitioned similarly and assume $(\mathbf{u}^{\mathrm{T}}, 1)L = 0$, $L(\mathbf{v}; 1) = 0$, where $\mathbf{u}, \mathbf{v}$ are $(n-1)$-vectors.

Then the inverse of the $(n-1) \times (n-1)$ matrix L_{11} exists and is given by

$$\begin{aligned} L_{11}^{-1} &= X \stackrel{\mathrm{def}}{=} (I_{n-1} + \mathbf{v}\mathbf{v}^{\mathrm{T}}) M_{11} (I_{n-1} + \mathbf{u}\mathbf{u}^{\mathrm{T}}) \\ &= (I_{n-1}\,,\, -\mathbf{v}) \begin{pmatrix} M_{11} & \mathbf{m}_{12} \\ \mathbf{m}_{21}^{\mathrm{T}} & m_{nn} \end{pmatrix} \begin{pmatrix} I_{n-1} \\ -\mathbf{u}^{\mathrm{T}} \end{pmatrix} \\ &= M_{11} - \mathbf{m}_{12}\mathbf{u}^{\mathrm{T}} - \mathbf{v}\mathbf{m}_{21}^{\mathrm{T}} + m_{nn}\mathbf{v}\mathbf{u}^{\mathrm{T}}. \end{aligned}$$

If $\mathbf{u} = \mathbf{v} = \mathbf{1}$ then $[L_{11}^{-1}]_{ij} = m_{ij} + m_{nn} - m_{in} - m_{nj}$.

Proof

- Idea: Plug prospective inverse X in to verify $XL_{11} = I$:

$$
\begin{aligned}
XL_{11} &= (I_{n-1}\,,\,-\mathbf{v})\begin{pmatrix} M_{11} & \mathbf{m}_{12} \\ \mathbf{m}_{21}^{\mathrm{T}} & m_{nn} \end{pmatrix}\begin{pmatrix} I_{n-1} \\ -\mathbf{u}^{\mathrm{T}} \end{pmatrix} L_{11} \\
&= (I_{n-1}\,,\,-\mathbf{v})\begin{pmatrix} M_{11} & \mathbf{m}_{12} \\ \mathbf{m}_{21}^{\mathrm{T}} & m_{nn} \end{pmatrix}\begin{pmatrix} L_{11} \\ \mathbf{l}_{21}^{\mathrm{T}} \end{pmatrix} && \boxed{\text{A}} \\
&= (I_{n-1}\,,\,-\mathbf{v})ML\begin{pmatrix} I_{n-1} \\ \mathbf{0}^{\mathrm{T}} \end{pmatrix} \\
&= (I_{n-1}\,,\,-\mathbf{v})\begin{pmatrix} I_{n-1} \\ \mathbf{0}^{\mathrm{T}} \end{pmatrix} = I_{n-1} && \boxed{\text{B}}
\end{aligned}
$$

$\boxed{\text{A}}$ From $(\mathbf{u}^{\mathrm{T}}, 1)L = (\mathbf{u}^{\mathrm{T}}L_{11} + \mathbf{l}_{21}^{\mathrm{T}}\ ,\ \mathbf{u}^{\mathrm{T}}\mathbf{l}_{12} + l_{nn}) = 0.$

$\boxed{\text{B}}$ From $ML = I_n - \begin{pmatrix} \mathbf{v} \\ 1 \end{pmatrix}(\mathbf{v}^T, 1)/(\mathbf{v}^{\mathrm{T}}\mathbf{v} + 1)$ (ortho projector).

Hitting and Commute Times

Adding up previous gives

- $\mathbf{H}(i,k) = \sum_j N(i,j,k) = m_{kk} - m_{ik} + \sum_j (m_{ij} - m_{kj})\pi_j$
- $\mathbf{C}(i,k) = \mathbf{H}(i,k) + \mathbf{H}(k,i) = m_{kk} + m_{ii} - m_{ik} - m_{ki}.$
- Above holds also for strongly connected directed graphs (arbitrary Markov chain with no transient states).
- Could add along other dimensions to get betweenness measures, etc.

Commute Times

- Pseudo inverse of $L = L^{\mathrm{a}}/14$ is a Gram matrix:

$$M = L^{+} = \frac{7}{90} \cdot \begin{pmatrix} \mathbf{83} & -1 & -37 & -43 & -19 & 17 \\ -1 & \mathbf{47} & -1 & -19 & -7 & -19 \\ -37 & -1 & \mathbf{83} & 17 & -19 & -43 \\ -43 & -19 & 17 & \mathbf{83} & -1 & -37 \\ -19 & -7 & -19 & -1 & \mathbf{47} & -1 \\ 17 & -19 & -43 & -37 & -1 & \mathbf{83} \end{pmatrix}$$

- $\Longrightarrow$ expected commute times in random walk $[(\ell_2 \text{ metric})^2]$

$$\mathbf{C} = \begin{bmatrix} \operatorname{diag}(L^{+}) \cdot \mathbf{1}^{\mathrm{T}} \\ + \mathbf{1} \cdot \operatorname{diag}(L^{+}) \\ - L^{+} - (L^{+})^{\mathrm{T}} \end{bmatrix} = \frac{14}{15} \cdot \begin{pmatrix} 0 & 11 & 20 & 21 & 14 & 11 \\ 11 & 0 & 11 & 14 & 9 & 14 \\ 20 & 11 & 0 & 11 & 14 & 21 \\ 21 & 14 & 11 & 0 & 11 & 20 \\ 14 & 9 & 14 & 11 & 0 & 11 \\ 11 & 14 & 21 & 20 & 11 & 0 \end{pmatrix}.$$

- Red numbers: average extra cost of detour thru given node.

Outline

- Graphs Matrices & Laplacians
- Average Hitting and Commute Times
- Embedding in Euclidean Space
- Electric Resistances
- Spectral Graph Partitioning: Cuts
- Cheeger-like bounds.
- Conclusions

Lemma 2 – Conditionally Definite

If M is a symmetric positive semi-definite Gram matrix of inner products,
Then $\mathbf{C} = \mathbf{d}_M \mathbf{1}^{\mathrm{T}} + \mathbf{1}\mathbf{d}_M^{\mathrm{T}} - 2M$ s.t. $c_{ij} = m_{ii} + m_{jj} - 2m_{ij}$ is the conditionally definite matrix of squared distances. [here $\mathbf{d}_M = (m_{11}; \ldots ; m_{nn})$]
Note "Conditionally definite" means $\mathbf{x}^{\mathrm{T}}\mathbf{C}\mathbf{x} \leq 0$ for all $\mathbf{x} \perp \mathbf{1}$, and for simplicity $c_{ii} = 0, \forall i$. A typical example is a matrix of pairwise squared ℓ_2 distances.

If $\mathbf{C}$ is a conditionally definite matrix,
Then one can find a matching semi-definite Gram matrix M.
Note: A prospective uncentered M is given by $2\widehat{M} = \mathbf{c}_k \mathbf{1}^{\mathrm{T}} + \mathbf{1}\mathbf{c}_k^{\mathrm{T}} - \mathbf{C}$, where $\mathbf{c}_k$ is some arbitrarily selected column out of $\mathbf{C}$.
The result can be centered around the origin, yielding:

$$M = \left(I - \frac{\mathbf{1}\mathbf{1}^{\mathrm{T}}}{n}\right) \widehat{M} \left(I - \frac{\mathbf{1}\mathbf{1}^{\mathrm{T}}}{n}\right) = -{}^1\!/_2 \left(I - \frac{\mathbf{1}\mathbf{1}^{\mathrm{T}}}{n}\right) \mathbf{C} \left(I - \frac{\mathbf{1}\mathbf{1}^{\mathrm{T}}}{n}\right).$$

[Schoenberg, 1935; Schoenberg, 1938; Berg et al., 1984; Gower & Legendre, 1986]

Proof: AWLOG $\mathbf{x}_1 = 0$. Then $c_{1k} = c_{k1} = \|x_k\|_2^2$.
So $c_{ij} = m_{ii} + m_{jj} - 2m_{ij} = c_{i1} + c_{1j} - 2m_{ij}$.

Embedding

- $L^+ = \mathbf{S}^T\mathbf{S}$ with

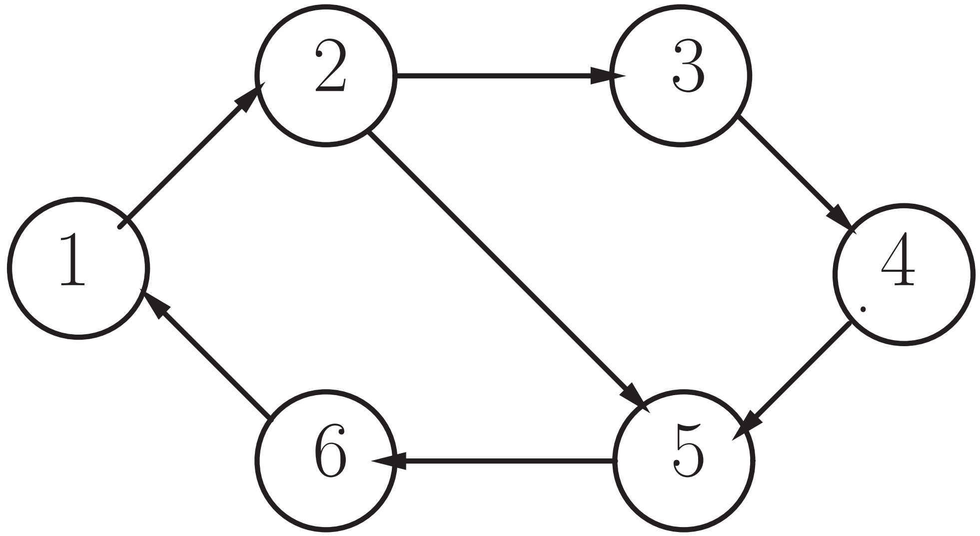
$$\mathbf{S} = \begin{pmatrix} \mathbf{s}_1 & \mathbf{s}_2 & \mathbf{s}_3 & \mathbf{s}_4 & \mathbf{s}_5 & \mathbf{s}_6 \\ 2.5408 & -.0306 & -1.1326 & -1.3163 & -.5816 & .52040 \\ 0 & 1.9117 & -.0588 & -.7941 & -.2941 & -.7647 \\ 0 & 0 & 2.2736 & -.0947 & -.9473 & -1.2315 \\ 0 & 0 & 0 & 2.02070 & -.5774 & -1.4434 \\ 0 & 0 & 0 & 0 & 1.4142 & -1.4142 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

- For all i, j, $\|\mathbf{s}_i - \mathbf{s}_j\|_2^2 = C_{ij}$.
- Since $L^+\mathbf{1} = \mathbf{0}$, the columns of $\mathbf{S}$ are already centered.

Outline

- Graphs Matrices & Laplacians
- Average Hitting and Commute Times *Directed Graph*
- Embedding in Euclidean Space
- Electric Resistances
- Spectral Graph Partitioning: Cuts
- Cheeger-like bounds.
- Conclusions

Example – Directed Graph



$$P = \begin{pmatrix} 0 & 1.0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.5 & 0 & 0.5 & 0 \\ 0 & 0 & 0 & 1.0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1.0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1.0 \\ 1.0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad \mathbf{d} = \begin{pmatrix} 1 \\ 2 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \quad \not\propto \quad \boldsymbol{\pi} = \begin{pmatrix} 0.2 \\ 0.2 \\ 0.1 \\ 0.1 \\ 0.2 \\ 0.2 \end{pmatrix}$$

Laplacian from Probabilities

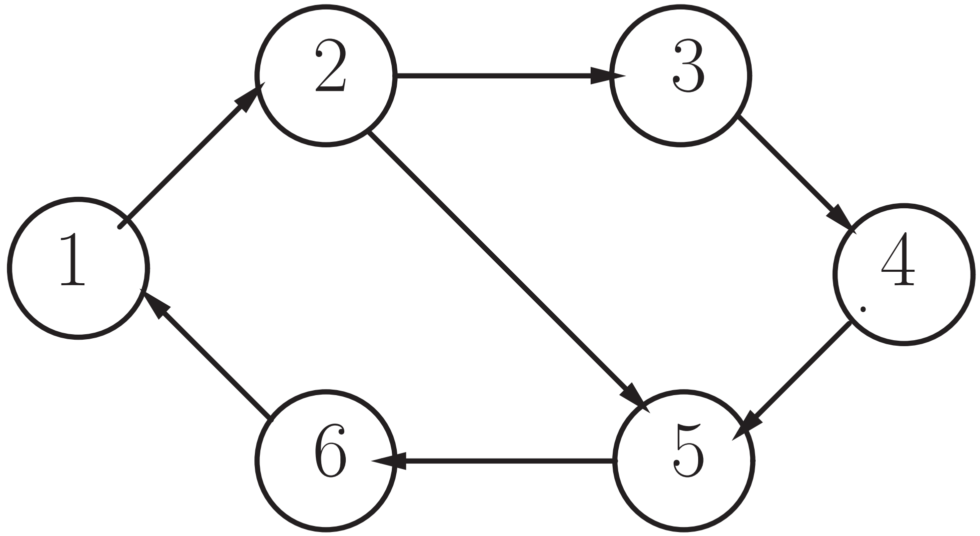
- Can still obtain commute times, but from $L = \Pi - \Pi P$:

$$L = \begin{pmatrix} 0.2 & -0.2 & 0 & 0 & 0 & 0 \\ 0 & 0.2 & -0.1 & 0 & -0.1 & 0 \\ 0 & 0 & 0.1 & -0.1 & 0 & 0 \\ 0 & 0 & 0 & 0.1 & -0.1 & 0 \\ 0 & 0 & 0 & 0 & 0.2 & -0.2 \\ -0.2 & 0 & 0 & 0 & 0 & 0.2 \end{pmatrix}, \begin{matrix}\text{null} \\ \text{vec}\end{matrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

$$M = L^+ = \frac{5}{6} \begin{pmatrix} \mathbf{3} & 2 & 0 & -2 & -1 & -2 \\ -2 & \mathbf{3} & 1 & -1 & 0 & -1 \\ -3 & -4 & \mathbf{6} & 4 & -1 & -2 \\ -1 & -2 & -4 & \mathbf{6} & 1 & 0 \\ 1 & 0 & -2 & -4 & \mathbf{3} & 2 \\ 2 & 1 & -1 & -3 & -2 & \mathbf{3} \end{pmatrix}$$

This Laplacian is only one with null vector $(1, \ldots, 1)$ on both sides.

Hitting & Commute Times



H (hitting times)

0	**1**	**6**	**7**	**3**	**4**
4	0	5	6	2	3
4	5	0	1	2	3
3	4	9	0	1	2
2	3	8	9	0	1
1	2	7	8	4	0

C (commute times)

0	**5**	**10**	**10**	**5**	**5**
5	0	10	10	5	5
10	10	0	10	10	10
10	10	10	0	10	10
5	5	10	10	0	5
5	5	10	10	5	0

- Only nodes 3, 4 are peripheral. Others are all equally important.
- Same reflected in average commute times from node 2.

Outline

- Graphs Matrices & Laplacians
- Average Hitting and Commute Times
- Embedding in Euclidean Space
- Electric Resistances
- Spectral Graph Partitioning: Cuts
- Cheeger-like bounds.
- Conclusions

Interpretations – Undirected Graphs

- Commute times correspond to effective resistances.
[Doyle & Snell, 1984; Chandra et al., 1989; Klein & Randic, 1993].

- Eigenvalues of

$$L^{\text{ds}} = I - P = \frac{1}{6} \cdot \begin{pmatrix} 6 & -3 & 0 & 0 & 0 & -3 \\ -2 & 6 & -2 & 0 & -2 & 0 \\ 0 & -3 & 6 & -3 & 0 & 0 \\ 0 & 0 & -3 & 6 & -3 & 0 \\ 0 & -2 & 0 & -2 & 6 & -2 \\ -3 & 0 & 0 & 0 & -3 & 6 \end{pmatrix}$$

are 0, $\boxed{1/2}$, 5/6, 7/6, 3/2, 2. The $\boxed{1/2}$ is related to the expander graph or Cheeger bound of the graph. [Chung, 2005; Zhou et al., 2005].

- Also $\boxed{1/2} \leftrightarrow$ mixing rate for random walk over the graph.

- The corresponding eigenvector used in spectral graph partitioning $(-1, 0, 1, 1, 0, -1)$.

Incidence Matrix

- The incidence matrix $\mathbf{N}$ has n columns and $vol(G)$ rows. Each column corresponds to a node (vertex) of graph G and each row corresponds to an edge (in some arbitrary order).

- The j-th row represents the edge $e_j = (i, j)$, and looks like

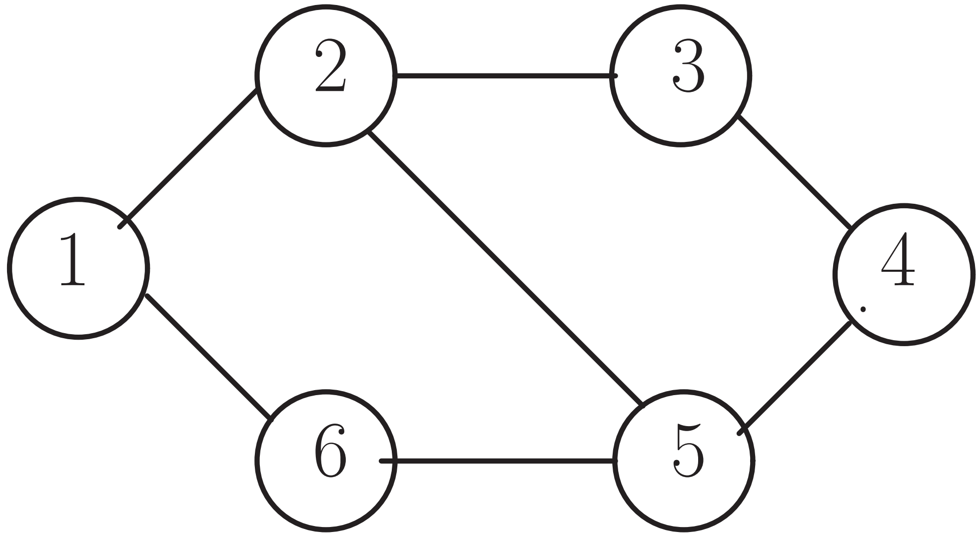
$$0, \ldots, 0, 1, 0, \ldots, 0, -1, 0, \ldots, 0$$

  where the nonzero entries are in columns i, j corresponding to the vertices connected by that edge.

- Then a simple calculation shows $L = D - A = \mathbf{N}^T\mathbf{N}$, where $A =$ adjacency matrix and $D =$ diagonal matrix of degrees.

- In general: if $\mathbf{v}$ is a vector of voltages, then $\mathbf{N}\mathbf{v}$ is the vector of currents across each link, assuming unit conductances.

Example Incidence Matrix



$$\mathbf{N} = \begin{pmatrix} -1 & +1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & +1 \\ 0 & +1 & 0 & 0 & -1 & 0 \\ 0 & +1 & -1 & 0 & 0 & 0 \\ 0 & 0 & +1 & -1 & 0 & 0 \\ 0 & 0 & 0 & +1 & -1 & 0 \\ 0 & 0 & 0 & 0 & +1 & -1 \end{pmatrix}$$

Resistances

[Doyle & Snell, 1984; Chandra et al., 1989; Klein & Randic, 1993].

- Current = Incidence_matrix · Voltage (using unit resistances):

$$\underset{\mathbf{I}}{\begin{pmatrix} \mathrm{i}_1 \\ \vdots \\ \mathrm{i}_7 \end{pmatrix}} = \underset{\mathbf{N}}{\begin{pmatrix} -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 \\ 1 & 0 & 0 & 0 & 0 & -1 \\ 0 & -1 & 0 & 0 & 1 & 0 \end{pmatrix}} \cdot \underset{\mathbf{v}}{\begin{pmatrix} \mathrm{v}_1 \\ \vdots \\ \mathrm{v}_6 \end{pmatrix}}$$

- Kirchoff's law: If unit current is injected between nodes i & j, then net current through every other vertex must be zero:

$$\mathbf{e}_i - \mathbf{e}_j = \mathbf{N}^T \mathbf{I} = \cdots = \mathbf{N}^T \mathbf{N} \mathbf{v} = L^{\mathrm{a}} \mathbf{v}.$$

- Solve for voltages = $\mathbf{v} = (L^{\mathrm{a}})^{+}(\mathbf{e}_i - \mathbf{e}_j)$.
- Net voltage drop i to j = effective resistance = $\mathrm{v}_i - \mathrm{v}_j = (\mathbf{e}_i - \mathbf{e}_j)^{\mathrm{T}} \mathbf{v} = (\mathbf{e}_i - \mathbf{e}_j)^{\mathrm{T}} (L^{\mathrm{a}})^{+} (\mathbf{e}_i - \mathbf{e}_j)$.

Resistances

$\mathbf{e}_i - \mathbf{e}_j \perp \text{Nullsp}(\mathbf{N}^T\mathbf{N})$, so can use pseudo-inverse to find voltages.

- Solve for voltages $\mathbf{v} = (\mathbf{N}^T\mathbf{N})^+ \cdot (\mathbf{e}_i - \mathbf{e}_j) = (L^{\mathrm{a}})^+(\mathbf{e}_i - \mathbf{e}_j)$.
- Effective resistance between nodes i & j is

$$\begin{aligned} \mathrm{v}_i - \mathrm{v}_j &= (\mathbf{e}_i^T - \mathbf{e}_j^T) \cdot \mathbf{v} \\ &= (\mathbf{e}_i^T - \mathbf{e}_j^T) \cdot (\mathbf{N}^T\mathbf{N})^+ \cdot (\mathbf{e}_i - \mathbf{e}_j) \\ &= (\mathbf{e}_i^T - \mathbf{e}_j^T) \cdot \quad (L^{\mathrm{a}})^+ \quad \cdot (\mathbf{e}_i - \mathbf{e}_j) \\ &= [(L^{\mathrm{a}})^+]_{ii} + [(L^{\mathrm{a}})^+]_{jj} - [(L^{\mathrm{a}})^+]_{ij} - [(L^{\mathrm{a}})^+]_{ji}. \end{aligned}$$

- Collect matrix of effective resistances: (= commute times)

$$\alpha\mathbf{C} = \text{diag}(L^{\mathrm{a}})^+ \cdot \mathbf{1}^{\mathrm{T}} + \mathbf{1} \cdot \text{diag}(L^{\mathrm{a}})^+ - (L^{\mathrm{a}})^+ - [(L^{\mathrm{a}})^+]^{\mathrm{T}}.$$

- The entries $\mathbf{C}_{ij}$ are squares of a Euclidean metric. [Schoenberg, 1935; Schoenberg, 1938; Berg et al., 1984],

Outline

- Graphs Matrices & Laplacians
- Average Hitting and Commute Times
- Embedding in Euclidean Space
- Electric Resistances
- Spectral Graph Partitioning: Cuts
- Cheeger-like bounds.
- Conclusions

Vector showing 2 classes

- Define $\mathbf{v} = \{\alpha, -\beta\}^n$ where $v_i = \alpha > 0$ is node i is in class A, and $v_i = -\beta < 0$ if node i is in class B.

- Then the non-zero entries of the vector $\mathbf{Nv}$ are in the positions corresponding to the edges with one end in class A and the other end in class B.

- Hence $\mathbf{v}^T\mathbf{N}^T\mathbf{N}\mathbf{v} = \mathbf{v}^T L\mathbf{v} = \mathrm{cut}(\mathrm{A}, \mathrm{B})(\alpha + \beta)^2$.

- Also $\mathbf{v}^T\mathbf{v} = n_{\mathrm{A}}\alpha^2 + n_{\mathrm{B}}\beta^2$.

- Also $\mathbf{v}^T D\mathbf{v} = d_{\mathrm{A}}\alpha^2 + d_{\mathrm{B}}\beta^2$

- Here $n_{\mathrm{A}} = \#$ vertices in class A, $d_{\mathrm{A}} =$ sum of all degrees of nodes in class A. Ditto for class B. And $n = n_{\mathrm{A}} + n_{\mathrm{B}} =$ total number of vertices, and $d = d_{\mathrm{A}} + d_{\mathrm{B}} = 2$ times total number of edges.

Cut relative to |nodes|

- Let $\alpha^2 = n_{\mathrm{B}}/n_{\mathrm{A}}$, $\beta^2 = n_{\mathrm{A}}/n_{\mathrm{B}}$.
- Then $\mathbf{v}^T L\mathbf{v} == \mathrm{cut}(\mathrm{A},\mathrm{B})\left(\frac{n_{\mathrm{A}}+n_{\mathrm{B}}}{\sqrt{n_{\mathrm{A}}n_{\mathrm{B}}}}\right)^2 = \mathrm{cut}(\mathrm{A},\mathrm{B})\frac{n^2}{n_{\mathrm{A}}n_{\mathrm{B}}}$,
- and $\mathbf{v}^T\mathbf{v} = n_{\mathrm{A}}(n_{\mathrm{B}}/n_{\mathrm{A}}) + n_{\mathrm{B}}(n_{\mathrm{A}}/n_{\mathrm{B}}) = n$.
- Hence
$$\frac{\mathbf{v}^T L\mathbf{v}}{\mathbf{v}^T\mathbf{v}} = \frac{\mathrm{cut}(\mathrm{A},\mathrm{B})}{n_{\mathrm{A}}n_{\mathrm{B}}}n$$
- Also $\mathbf{v}^T\mathbf{1} = n_{\mathrm{A}}\alpha - n_{\mathrm{B}}\beta = \sqrt{n_{\mathrm{A}}n_{\mathrm{B}}} - \sqrt{n_{\mathrm{B}}n_{\mathrm{A}}} = 0$.
- Hence
$$\frac{\mathbf{v}^T L\mathbf{v}}{\mathbf{v}^T\mathbf{v}} \geq \min_{\mathbf{x}\perp\mathbf{1}} \frac{\mathbf{x}^T L\mathbf{x}}{\mathbf{x}^T\mathbf{x}}.$$

Cut relative to |edges|

- Now look at minimal cut relative to the number of edges in each half.
- Let $\alpha^2 = d_\mathrm{B}/d_\mathrm{A}$, $\beta^2 = d_\mathrm{A}/d_\mathrm{B}$.
- Then $\mathbf{v}^T L \mathbf{v} == \mathrm{cut}(\mathrm{A},\mathrm{B}) \left(\frac{d_\mathrm{A} + d_\mathrm{B}}{\sqrt{d_\mathrm{A} d_\mathrm{B}}} \right)^2 = \mathrm{cut}(\mathrm{A},\mathrm{B}) \frac{d^2}{d_\mathrm{A} d_\mathrm{B}}$,
- and $\mathbf{v}^T D \mathbf{v} = d_\mathrm{A}(d_\mathrm{B}/d_\mathrm{A}) + d_\mathrm{B}(d_\mathrm{A}/d_\mathrm{B}) = d$.
- Hence

$$\frac{\mathbf{v}^T L \mathbf{v}}{\mathbf{v}^T D \mathbf{v}} = \frac{\mathrm{cut}(\mathrm{A},\mathrm{B})}{d_\mathrm{A} d_\mathrm{B}} d$$

Generalized Eigenvalue Problem

- Let $\mathbf{w} = D^{1/2}\mathbf{v}$. Then $\mathbf{w}^T\sqrt{\mathbf{d}} = \mathbf{v}^T\mathbf{d} = \alpha d_{\mathrm{A}} - \beta d_{\mathrm{B}} = 0$.

- Also $\mathcal{L}\sqrt{\mathbf{d}} = D^{-1/2}LD^{-1/2}\sqrt{\mathbf{d}} = 0$.

- The Rayleigh Quotient is

$$\frac{\mathbf{v}^T L\mathbf{v}}{\mathbf{v}^T D\mathbf{v}} = \frac{\mathbf{w}^T \mathcal{L}\mathbf{w}}{\mathbf{w}^T\mathbf{w}} \geq \min_{\mathbf{x}\perp\sqrt{\mathbf{d}}} \frac{\mathbf{x}^T\mathcal{L}\mathbf{x}}{\mathbf{x}^T\mathbf{x}} = \lambda_2(\mathcal{L}).$$

Relation to Random Walk

- The smallest non-zero eigenvalue of $\mathcal{L}$ is related to best edge-relative cut.
- The eigenvalues of $\mathcal{L}$ are the same as the eigenvalues of $I - P$:

$$D^{-1/2}\mathcal{L}D^{1/2} = D^{-1/2}(I - D^{-1/2}AD^{-1/2})D^{1/2} = I - D^{-1}A = I - P.$$

- The smallest non-zero eigenvalue of $\mathcal{L}$ corresponds to second largest eigenvalue of P, i.e., the mixing rate.
- The largest eigenvalue of $\mathcal{L}$ corresponds to the smallest (most negative) eigenvalue of P. The latter is at least -1 (exactly -1 iff random walk is 2-cyclic, periodic). So the former is at most 2, and exactly equal to 2 iff graph is bipartite.

Outline

- Graphs Matrices & Laplacians
- Average Hitting and Commute Times
- Embedding in Euclidean Space
- Electric Resistances
- Spectral Graph Partitioning: Cuts
- Cheeger-like bounds.
- Conclusions

Cheeger Bounds

- Denote the eigenvalue of $\mathcal{L}$ as $0 = \lambda_1 < \lambda_2 \leq \cdots \leq \lambda_n \leq 2$.
- The basic Cheeger bound is [Chung, 2005]

$$2h_G \geq \lambda_2(\mathcal{L}) \geq {}^1\!/_2 h_G^2$$

where

h_G = minimum cut relative to the edge weights,

$\lambda_2(\mathcal{L})$ = 2nd smallest eigenvalue of $\mathcal{L} = I - D^{-1/2} A D^{-1/2}$.

Isoperimetric Constant

Definitions: [Chung, 2005]

- Neighborhood of set X of nodes, $N(X)$, is the set of nodes not in X but with an edge to X.
- $g_G = \min_{X: vol(X) \leq vol(\overline{X})} \frac{vol(N(X))}{vol(X)}$

2 bounds: [Chung, 2005]

- $\lambda_2 \geq \frac{g_G^2}{2d(2 + 2g_G + g_G^2)}$.
- $g_G \geq \frac{1-(1-\lambda')^2}{(1-\lambda')^2 + \frac{vol(X)}{vol(\overline{X})}} \geq (1-(1-\lambda')^2)(1 - \frac{vol(X)}{vol(\overline{X})}$,

  where $\lambda' = \frac{2\lambda_2}{\lambda_2 + \lambda_n}$ if $1 - \lambda_2 < \lambda_n - 1$, and $\lambda' = \lambda_2$ o.w.

Conclusions

- Introduced Laplacian for undirected graph
- Laplacian Related to Average Commute Times
- Laplacian Related to Electric Resistance
- Laplacian Related to mixing times and Graph Cuts.

References

Bamieh, B., Jovanovic, M., Mitra, P., & Patterson, S. (2008). Effect of topological dimension on rigidity of vehicle formations: Fundamental limitations of local feedback. *Proc. CDC* (pp. 369–374). Cancun, Mexico.

Berg, C., Christiansen, J., & Ressel, P. (1984). *Harmonic analysis on semigroups.* Springer Verlag.

Boley, D., Ranjan, G., & Zhang, Z.-L. (2010). Commute times for a directed graph using an asymmetric laplacian. U of Mn CS&E Dept report TR 10-005, Univ. of Minn., Dept. of Comp uter Science and Eng. (a revised version is to appear in LAA 2011).

Boley, D., Ranjan, G., & Zhang, Z.-L. (2011). Commute times for a directed graph using an asymmetric Laplacian. *Linear Algebra and Appl.*, *435*, 224–242.

Brualdi, R. A., & Ryser, H. J. (1991). *Combinatorial matrix theory.* Cambridge Univ. Press.

Chandra, A., Raghavan, P., Ruzzo, W., Smolensky, R., & Tiwari, P. (1989). The electrical resistance of a graph captures its commute and cover times. *Proc. of Annual ACM Symposium on Theory of Computing* (pp. 574–586).

Chebotarev, P., & Shamis, E. (2006). Matrix-forest theorems.

Chung, F. (2005). Laplacians and the Cheeger inequality for directed graphs. *Annals of Combinatorics*, *9*, 1–19.

Doyle, P., & Snell, J. (1984). *Random walks and electric networks.* The Math. Assoc. of Am. `front.math.ucdavis.edu/math.PR/0001057`.

Fouss, F., Pirotte, A., Renders, J., & Saerens, M. (2007). Random-walk computation of similarities between nodes of a graph with application to collaborative recommendation. *IEEE Trans. on Knowledge and Data Engineering*, *19*, 355–369.

Gower, J., & Legendre, P. (1986). Metric and euclidean properties of dissimilarities coefficients. *J. Classification*, *3*, 5–48.

Grinstead, C. M., & Snell, J. L. (2006). *Introduction to probability*. American Mathematical Society. 2nd edition, `www.dartmouth.edu/~chance/teaching_aids/books_articles/probability_book/book.html`.

Klein, D., & Randic, M. (1993). Resistance distance. *J. Math. Chemistry*, *12*, 81–95.

Olfati-Saber, R., Murray, R. M., A, & B (2004). Consensus problems in networks of agents with switching topology and time-delays. *IEEE Trans. Auto. Contr.*, *49*, 1520–1533.

Schoenberg, I. J. (1935). Remarks to Maurice Fréchet's article "Sur la définition axiomatique d'une classe d'espace distanciés vectoriellement applicable sur l'espace de Hilbert". *Annals of Mathematics*, *36*, 724–732.

Schoenberg, I. J. (1938). Metric spaces and positive definite functions. *Trans. of the Amer. Math Soc.*, *44*, 522–536.

Shi, J., & Malik, J. (2000). Normalised cuts and image segmentation. *IEEE Trans. Pattern Analysis and Machine Intelligence*, *22*, 888–905.

Spielman, D. A., & Teng, S.-H. (1996). Spectral partitioning works:planar graphs and finite element meshes. *37th Annual Symposium on Foundations of Computer Science*. IEEE Computer Soc. Press.

von Luxburg, U. (2007). A tutorial on spectral clustering. *Statistics and Computing*, *17*, 395–416. Max Planck Institute for Biological Cybernetics. Technical Report No. TR-149.

Young, G. F., Scandovi, L., & Leonard, N. (2010). Robustness of noisy consensus dynamics with directed communication. *Proc. ACC* (pp. 6312–6317).

Zhou, D., Huang, J., & Schölkopf, B. (2005). Learning from labeled and unlabeled data on a directed graph. *Proc. 22nd Int'l Conf. Machine Learning* (pp. 1041–1048).