Laplacians of graphs and Cheeger inequalities

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Outline

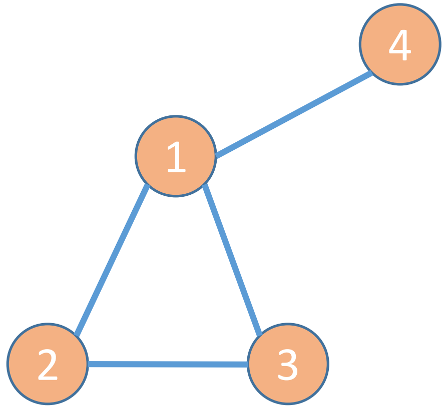
- Review on Graph Analysis
  - Combinatorial Laplacian
  - Normalized Laplacian
  - Cut
- Cheeger Constant
- Cheeger Inequality for Edge Expansion
  - Proof
  - Examples
- Isoperimetric Inequality for Vertex Expansion

- A graph is an ordered pair $G = (V, E)$
- Adjacency matrix

$$A_{uv} = \begin{cases} 1 & \text{if } (u,v) \in E(G) \\ 0 & \text{otherwise} \end{cases}$$

  Undirected graph $\Longleftrightarrow$ symmetric adjacency matrix

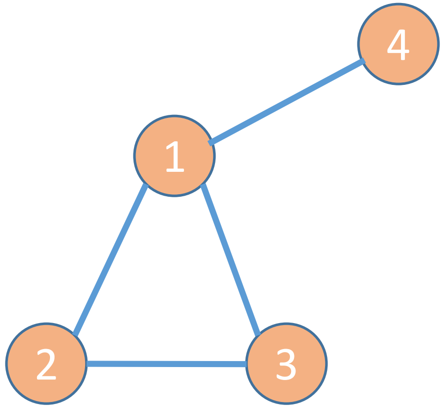
- d_v: degree of the vertex v
- $D = diag(d)$ = diagonal matrix of degrees



$$A = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \; d = \begin{pmatrix} 3 \\ 2 \\ 2 \\ 1 \end{pmatrix}, \; D = \begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Combinatorial Laplacian

$$L(u,v) = \begin{cases} d_v & \text{if } u = v \\ -1 & \text{if } u \text{ and } v \text{ are adjacent} \\ 0 & \text{otherwise} \end{cases}$$



$$L = D - A = \begin{pmatrix} 3 & -1 & -1 & -1 \\ -1 & 2 & -1 & 0 \\ -1 & -1 & 2 & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix}$$

Normalized Laplacian

$$\mathcal{L} = SLS = D^{-\frac{1}{2}}\, L\, D^{-\frac{1}{2}}$$

$S = D^{-\frac{1}{2}}$: diagonal matrix with the (v, v)-th entry having value $\frac{1}{\sqrt{d_v}}$

$$\mathcal{L}(u,v) = \begin{cases} 1 & \text{if } u = v \\ -\dfrac{1}{\sqrt{d_u d_v}} & \text{if } u \text{ and } v \text{ are adjacent} \\ 0 & \text{otherwise} \end{cases}$$

with eigenvalues

$$0 = \lambda_1 \le \lambda_2 \le \cdots \le \lambda_n \le 2$$

The smallest non-zero eigenvalue of $\mathcal{L}$ is related to best edge-relative cut.

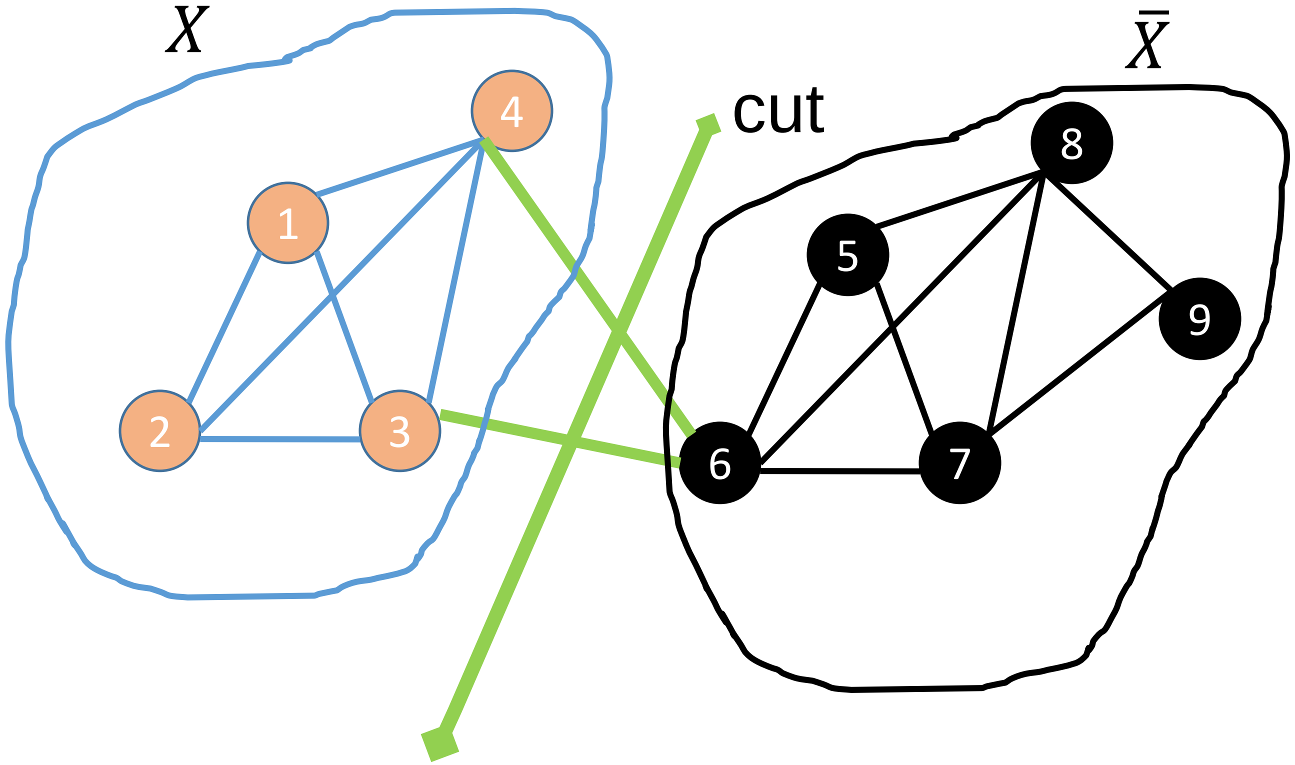
Cut: results in disconnected graph

- Vertex cut
- Edge cut

How easy is it to cut a graph?

Efficient Cut:

- Should be small
- Pieces left to be large

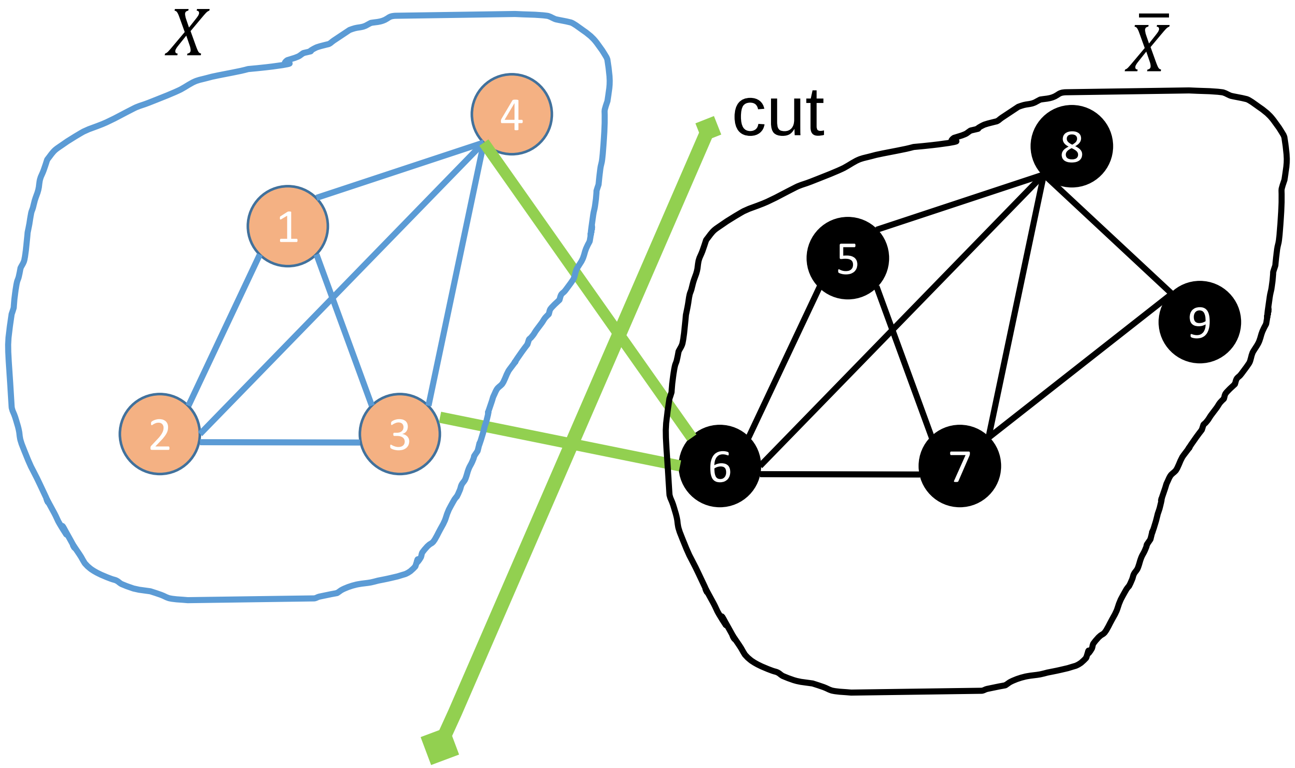


Cheeger constant h_G

For a subset $X \subset V$

$$h_G(X) = \frac{|E(X,\bar{X})|}{\min(\sum_{x\in X} d_x, \sum_{y\in \bar{X}} d_y)}$$

$$h_G = \min_X h_G(X)$$



$E(X,\bar{X})$: denotes the set of edges with one vertex in X and one vertex in $\bar{X}$.

$$\sum_{x\in X} d_x = vol(X)$$

If G is disconnected, h_G=0

If G is connected, $h_G > 0$

Cheeger Inequalities: $2h_G \geq \lambda_2 \geq \frac{h_G^2}{2}$

1. $\lambda_2 \leq 2h_G$

Let h denote a function which assigns to each vertex v of G a complex value $h(v)$. Then

$$\lambda_2 = \min_{f \perp S^{-2}\mathbf{1}} \frac{\sum_{u \sim v} (f(u) - f(v))^2}{\sum_v d_v f(v)^2}$$

where $h = S^{-1} f$.

We choose f based on an optimum cut C which achieves h_G and separates the graph G into two parts, A and B:

$$f(v) = \begin{cases} \frac{1}{vol(A)} & \text{if } v \text{ is in } A \\ -\frac{1}{vol(B)} & \text{if } v \text{ is in } B \end{cases}$$

By substituting f into λ_2, we have the following:

$$\begin{aligned} \lambda_2 &\leq |C|(1/vol(A) + 1/vol(B)) \\ &\leq \frac{2|C|}{\min(vol(A), vol(B))} \\ &= 2h_G \end{aligned}$$

2. $\lambda_2 \geq \frac{h_G^2}{2}$

PROOF. We consider the harmonic eigenfunction f of $\mathcal{L}$ with eigenvalue λ_2 We order vertices of G according to f. That is, relabel the vertices so that $f(v_i) \geq f(v_{i+1})$, for $1 \leq i \leq n-1$. Let $S_i = \{v_1, \ldots, v_i\}$ and define

$$\alpha_G = \min_i h_{S_i}.$$

Let r denote the largest integer such that $\mathrm{vol}(S_r) \leq \mathrm{vol}(G)/2$. Since $\sum_v g(v) d_v = 0$,

$$\sum_v g(v)^2 d_v = \min_c \sum_v (g(v) - c)^2 d_v \leq \sum_v (g(v) - g(v_r))^2 d_v.$$

We define the positive and negative part of $g - g(v_r)$, denoted by g_+ and g_-, respectively, as follows:

$$g_+(v) = \begin{cases} g(v) - g(v_r) & \text{if } g(v) \geq g(v_r), \\ 0 & \text{otherwise,} \end{cases}$$

$$g_-(v) = \begin{cases} |g(v) - g(v_r)| & \text{if } g(v) \leq g(v_r), \\ 0 & \text{otherwise.} \end{cases}$$

We consider

$$\begin{aligned} \lambda_G &= \frac{\sum_{u \sim v} (g(u) - g(v))^2}{\sum_v g(v)^2 d_v} \\ &\geq \frac{\sum_{u \sim v} (g(u) - g(v))^2}{\sum_v (g(v) - g(v_r))^2 d_v} \\ &\geq \frac{\sum_{u \sim v} \left((g_+(u) - g_+(v))^2 + (g_-(u) - g_-(v))^2 \right)}{\sum_v \left(g_+(v)^2 + g_-(v)^2 \right) d_u}. \end{aligned}$$

Without loss of generality, we assume $R(g_+) \leq R(g_-)$ and therefore we have $\lambda_G \geq R(g_+)$ since

$$\frac{a+b}{c+d} \geq \min\{\frac{a}{c}, \frac{b}{d}\}.$$

We here use the notation

$$\tilde{\mathrm{vol}}(S) = \min\{\mathrm{vol}(S), \mathrm{vol}(G) - \mathrm{vol}(S)\}$$

so that

$$|\partial(S_i)| \geq \alpha_G \tilde{\mathrm{vol}}(S_i).$$

Then we have

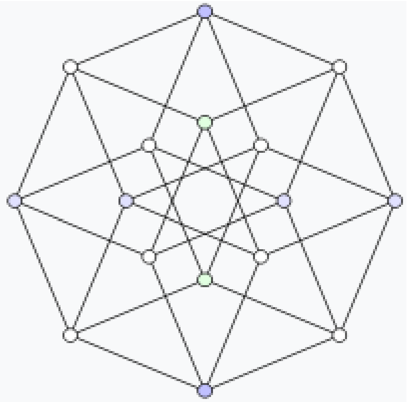
$$\begin{aligned}
\lambda_G &\geq R(g_+) \\
&= \frac{\sum_{u\sim v}(g_+(u) - g_+(v))^2}{\sum_u g_+^2(u)d_u} \\
&= \frac{\big(\sum_{u\sim v}(g_+(u) - g_+(v))^2\big)\big(\sum_{u\sim v}(g_+(u) + g_+(v))^2\big)}{\sum_u g_+^2(u)d_u \sum_{u\sim v}(g_+(u) + g_+(v))^2} \\
&\geq \frac{\big(\sum_i |g_+(v_i)^2 - g_+(v_{i+1})^2|\alpha_G|\tilde{\mathrm{vol}}(S_i)|\big)^2}{2\big(\sum_u g_+^2(u)d_u\big)^2} \\
&= \frac{\alpha_G^2}{2}\frac{\big(\sum_i g_+(v_i)^2(|\tilde{\mathrm{vol}}(S_i) - \tilde{\mathrm{vol}}(S_{i+1})|)\big)^2}{\big(\sum_u g_+^2(u)d_u\big)^2} \\
&= \frac{\alpha_G^2}{2}\frac{\big(\sum_i g_+(v_i)^2 d_{v_i}\big)^2}{\big(\sum_u g_+^2(u)d_u\big)^2} \\
&= \frac{\alpha_G^2}{2}.
\end{aligned}$$

Examples

- Hypercube graph Q_n : the graph formed from the vertices and edges of an n-dimensional hypercube

$$\lambda = \frac{2k}{n}, \text{ for } k = 0,1,\ldots,n$$

$$h_{Q_n} = \frac{1}{n}$$



The hypercube graph Q_4

- Path graph P_n

$$\lambda = 1 - \cos\frac{k\pi}{n-1}, \text{ for } k = 0,1,\ldots,n-1$$

$$h_{P_n} = \frac{1}{n-1}$$

A path graph on 6 vertices

Isoperimetric Inequality for Vertex Expansion

For a subset X of vertices of V, we consider

$$N(X) = \{v \notin X : v \sim u \in X\}.$$

We define

$$g_G(X) = \frac{vol(N(X))}{\min(vol(X), vol(\bar{X}))}$$

and

$$g_G = \min_X g_G(X)$$

For a graph G,

$$\lambda_G \geq \frac{g_G^2}{2d(2 + 2g_G + g_G^2)}$$

where d denotes the maximum degree of G.

References

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- Fan Chung, The PageRank of a graph

Thank You!