✍12 Complexity? [number of multiplications and additions]

Solution: Let $\boldsymbol{A} \in \mathbb{R}^{m \times n}$ and $\boldsymbol{B} \in \mathbb{R}^{n \times p}$. Then the product $\boldsymbol{AB}$ requires $\boldsymbol{2mnp}$ operations (there are $\boldsymbol{mp}$ entries in all and each of them requires $\boldsymbol{2n}$ operations).

✍15 Show that $\boldsymbol{A} \in \mathbb{R}^{n \times n}$ is of rank one iff [if and only if] there exist two nonzero vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ such that

$$\boldsymbol{A} = \boldsymbol{u}\boldsymbol{v}^T.$$

What are the eigenvalues and eigenvectors of $\boldsymbol{A}$?

Solution: When both $\boldsymbol{u}$ and $\boldsymbol{v}$ are nonzero vectors then the rank of a matrix of the matrix $\boldsymbol{A} = \boldsymbol{u}\boldsymbol{v}^T$ is one. The range of $\boldsymbol{A}$ is the set of all vectors of the form

$$\boldsymbol{y} = \boldsymbol{A}\boldsymbol{x} = \boldsymbol{u}\boldsymbol{v}^T\boldsymbol{x} = (\boldsymbol{v}^T\boldsymbol{x})\boldsymbol{u}$$

since $\boldsymbol{u}$ is a nonzero vector, and not all vectors $\boldsymbol{v}^T\boldsymbol{x}$ are zero (because $\boldsymbol{v} \neq \boldsymbol{0}$) then this space is of dimension 1.

Eigenvalues /vectors

Write $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{\lambda}\boldsymbol{x}$ then notice that this means $(\boldsymbol{v}^T\boldsymbol{x})\boldsymbol{u} = \boldsymbol{\lambda}\boldsymbol{x}$ so either $\boldsymbol{v}^T\boldsymbol{x} = \boldsymbol{0}$ and $\boldsymbol{\lambda} = \boldsymbol{0}$ or $\boldsymbol{x} = \boldsymbol{u}$ and $\boldsymbol{\lambda} = \boldsymbol{v}^T\boldsymbol{u}$. Two eigenvalues: 0 and $\boldsymbol{v}^T\boldsymbol{x}$...

✍16 Is it true that

$$\mathbf{rank}(\boldsymbol{A}) = \mathbf{rank}(\bar{\boldsymbol{A}}) = \mathbf{rank}(\boldsymbol{A}^{\boldsymbol{T}}) = \mathbf{rank}(\boldsymbol{A}^{\boldsymbol{H}}) \text{ ?}$$

Solution:

The answer is yes and it follows from the fact that the ranks of $\boldsymbol{A}$ and $\boldsymbol{A}^{\boldsymbol{T}}$ are the same and the ranks of $\boldsymbol{A}$ and $\bar{\boldsymbol{A}}$ are also the same.

It is known that $\boldsymbol{rank}(\boldsymbol{A}) = \boldsymbol{rank}(\boldsymbol{A}^{\boldsymbol{T}})$. We now compare the ranks of $\boldsymbol{A}$ and $\bar{\boldsymbol{A}}$ (everything is considered to be complex).

The important property that is used is that if a set of vectors is linearly independent then so is its conjugate. [convince yourself of this by looking at material from 2033]. If $\boldsymbol{A}$ has rank $\boldsymbol{r}$ and for example its first $\boldsymbol{r}$ columns are the basis of the range, the the same $\boldsymbol{r}$ columns of $\bar{\boldsymbol{A}}$ are also linearly independent. So $\boldsymbol{rank}(\bar{\boldsymbol{A}}) \geq \boldsymbol{rank}(\boldsymbol{A})$. Now you can use a similar argument to show that $\boldsymbol{rank}(\boldsymbol{A}) \geq \boldsymbol{rank}(\bar{\boldsymbol{A}})$. Therefore the ranks are the same.

✍21 Eigenvalues of $\boldsymbol{A}$ and $\boldsymbol{B}$ are the same. What about eigenvectors?

Solution: If $\boldsymbol{Au} = \boldsymbol{\lambda u}$ then $\boldsymbol{XBX^{-1}u} = \boldsymbol{\lambda u} \rightarrow \boldsymbol{B(X^{-1}u)} = \boldsymbol{\lambda(Xu)} \rightarrow \boldsymbol{\lambda}$ is an eigenvalue of $\boldsymbol{B}$ with eigenvector $\boldsymbol{Xu}$ (note the $\boldsymbol{Xu}$ cannot be equal to zero because $\boldsymbol{u} \neq \boldsymbol{0}$ and $\boldsymbol{X}$ is nonsingular)