✍2 If $\boldsymbol{A} \in \mathbb{R}^{m\times n}$ what are the dimensions of $\boldsymbol{A}^\dagger$?, $\boldsymbol{A}^\dagger\boldsymbol{A}$?, $\boldsymbol{A}\boldsymbol{A}^\dagger$?

Solution: The dimension of $\boldsymbol{A}^\dagger\boldsymbol{A}$ is $\boldsymbol{n} \times \boldsymbol{m}$ and so $\boldsymbol{A}^\dagger\boldsymbol{A}$? is of size $\boldsymbol{n} \times \boldsymbol{n}$. Similarly, $\boldsymbol{A}\boldsymbol{A}^\dagger$ is of size $\boldsymbol{m} \times \boldsymbol{m}$.□

✍3 Show that $\boldsymbol{A}^\dagger\boldsymbol{A}$ is an orthogonal projector. What are its range and null-space?

Solution: One way to do this is to use the rank-one expansion: $\boldsymbol{A} = \sum \boldsymbol{\sigma}_i \boldsymbol{u}_i \boldsymbol{v}_i^T$. Then $\boldsymbol{A}^\dagger = \sum \frac{1}{\boldsymbol{\sigma}_i} \boldsymbol{v}_i \boldsymbol{u}_i^T$ and therefore,

$$\boldsymbol{A}^\dagger\boldsymbol{A} = \left[\sum_{i=1}^{r} \frac{1}{\boldsymbol{\sigma}_i} \boldsymbol{v}_i \boldsymbol{u}_i^T\right] \times \left[\sum_{j=1}^{r} \boldsymbol{\sigma}_j \boldsymbol{u}_j \boldsymbol{v}_j^T\right] = \sum_{j=1}^{r} \boldsymbol{v}_j \boldsymbol{v}_j^T$$

which is a projector. □

✍4 Consider the matrix:

$$\boldsymbol{A} = \begin{pmatrix} \mathbf{1} & \mathbf{0} & \mathbf{2} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{-2} & \mathbf{1} \end{pmatrix}$$

- Compute the singular value decomposition of $\boldsymbol{A}$

Find the SVD of $\boldsymbol{A}$ ...

Solution: The nonzero singular values of $\boldsymbol{A}$ are the square roots of the eigenvalues of

$$\boldsymbol{A}\boldsymbol{A}^{\boldsymbol{T}} = \begin{pmatrix} \mathbf{5} & \mathbf{-4} \\ \mathbf{-4} & \mathbf{5} \end{pmatrix}$$

These eigenvalues are $\mathbf{5} \pm \mathbf{4}$ and so $\boldsymbol{\sigma}_{\mathbf{1}} = \mathbf{3}, \boldsymbol{\sigma}_{\mathbf{2}} = \mathbf{1}$.

The matrix $\boldsymbol{U}$ of the left singular vectors is the matrix

$$\boldsymbol{U} = \frac{\mathbf{1}}{\sqrt{\mathbf{2}}} \begin{pmatrix} \mathbf{1} & \mathbf{1} \\ \mathbf{-1} & \mathbf{1} \end{pmatrix}$$

If $\boldsymbol{A} = \boldsymbol{U}\boldsymbol{\Sigma}\boldsymbol{V}^{\boldsymbol{T}}$, then $\boldsymbol{U}' * \boldsymbol{A} = \boldsymbol{\Sigma}\boldsymbol{V}^{\boldsymbol{T}}$. Therefore to get $\boldsymbol{V}$ we use the relation: $\boldsymbol{V}^{\boldsymbol{T}} =$

$\boldsymbol{\Sigma}^{-1} * \boldsymbol{U'} * \boldsymbol{A}$. We have

$$\boldsymbol{U'} * \boldsymbol{A} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 4 & -1 \\ 1 & 0 & 0 & 1 \end{pmatrix} \to \boldsymbol{V}^T = \frac{1}{\sqrt{2}} \begin{pmatrix} 1/3 & 0 & 4/3 & -1/3 \\ 1 & 0 & 0 & 1 \end{pmatrix} \to$$

□

- Find the matrix $\boldsymbol{B}$ of rank 1 which is the closest to $\boldsymbol{A}$ in 2-norm sense.

Solution: This is obtained by setting $\boldsymbol{\sigma_2}$ to zero in the SVD - or - equivalently as $\boldsymbol{B} = \boldsymbol{\sigma_1 u_1 v_1^T}$.

You will find

$$\boldsymbol{B} = \begin{pmatrix} 1/2 & 0 & 2 & -1/2 \\ -1/2 & 0 & -2 & 1/2 \end{pmatrix}$$

□