Inner products and Norms

Inner product of 2 vectors

➤ Inner product of 2 vectors x and y in $\mathbb{R}^n$:

$$x_1y_1 + x_2y_2 + \cdots + x_ny_n \text{ in } \mathbb{R}^n$$

Notation: (x, y) or y^Tx

➤ For complex vectors

$$(x, y) = x_1\bar{y}_1 + x_2\bar{y}_2 + \cdots + x_n\bar{y}_n \text{ in } \mathbb{C}^n$$

Note: $(x, y) = y^Hx$

Properties of Inner Product:

- $(x, y) = \overline{(y, x)}$.
- $(\alpha x + \beta y, z) = \alpha(x, z) + \beta(y, z)$ [Linearity]
- $(x, x) \geq 0$ is always real and non-negative.
- $(x, x) = 0$ iff $x = 0$ (for finite dimensional spaces).
- Given $A \in \mathbb{C}^{m \times n}$ then

$$(Ax, y) = (x, A^H y) \quad \forall\, x \in \mathbb{C}^n, \forall y \in \mathbb{C}^m$$

Vector norms

Norms are needed to measure lengths of vectors and closeness of two vectors. Examples of use: Estimate convergence rate of an iterative method; Estimate the error of an approximation to a given solution; ...

➤ A vector norm on a vector space $\mathbb{X}$ is a real-valued function on $\mathbb{X}$, which satisfies the following three conditions:

1. $\|x\| \geq 0, \quad \forall\, x \in \mathbb{X},$ and $\|x\| = 0$ iff $x = 0$.
2. $\|\alpha x\| = |\alpha|\|x\|, \quad \forall\, x \in \mathbb{X}, \quad \forall \alpha \in \mathbb{C}$.
3. $\|x + y\| \leq \|x\| + \|y\|, \quad \forall\, x, y \in \mathbb{X}$.

➤ Third property is called the triangle inequality.

Important example: Euclidean norm on $\mathbb{X} = \mathbb{C}^n$,

$$\|x\|_2 = (x, x)^{1/2} = \sqrt{|x_1|^2 + |x_2|^2 + \ldots + |x_n|^2}$$

✍1 Show that when Q is orthogonal then $\|Qx\|_2 = \|x\|_2$

➤ Most common vector norms in numerical linear algebra: special cases of the Hölder norms (for $p \geq 1$):

$$\|x\|_p = \left(\sum_{i=1}^{n} |x_i|^p \right)^{1/p}.$$

✍2 Find out (bbl search) how to show that these are indeed norms for any $p \geq 1$ (Not easy for 3rd requirement!)

Property: ➤ Limit of $\|x\|_p$ when $p \to \infty$ exists:

$$\lim_{p\to\infty} \|x\|_p = \max_{i=1}^{n} |x_i|$$

➤ Defines a norm denoted by $\|.\|_\infty$.

➤ The cases $p = 1$, $p = 2$, and $p = \infty$ lead to the most important norms $\|.\|_p$ in practice. These are:

$$\|x\|_1 = |x_1| + |x_2| + \cdots + |x_n|,$$
$$\|x\|_2 = \left[|x_1|^2 + |x_2|^2 + \cdots + |x_n|^2\right]^{1/2},$$
$$\|x\|_\infty = \max_{i=1,\ldots,n} |x_i|.$$

➤ The Cauchy-Schwartz inequality (important) is:

$$|(x,y)| \le \|x\|_2 \|y\|_2.$$

✍3 When do you have equality in the above relation?

✍4 Expand $(x+y, x+y)$. What does the Cauchy-Schwarz inequality imply?

➤ The Hölder inequality (less important for $p \neq 2$) is:

$$|(x,y)| \le \|x\|_p \|y\|_q \text{ , with } \frac{1}{p} + \frac{1}{q} = 1$$

✍5 Second triangle inequality: $| \, \|x\| - \|y\| \, | \le \|x - y\|$.

✍6 Consider the metric $d(x,y) = max_i |x_i - y_i|$. Show that any norm in $\mathbb{R}^n$ is a continuous function with respect to this metric.

Solution: We need to show that we can make $\|y\|$ arbitrarily close to $\|x\|$ by making y 'close' enough to x, where 'close' is measured in terms of the infinity norm distance $d(x,y) = \|x-y\|_\infty$. Define $u = x - y$ and write u in the canonocal basis as $u = \sum_{i=1}^n \delta_i e_i$. Then:

$$\|u\| = \|\sum_{i=1}^{n} \delta_i e_i\| \le \sum_{i=1}^{n} |\delta_i| \; \|e_i\| \le \max |\delta_i| \sum_{i=1}^{n} \|e_i\|$$

Setting $M = \sum_{i=1}^n \|e_i\|$ we get $\boxed{\|u\| \le M \max |\delta_i| = M\|x-y\|_\infty}$

Let ϵ be given and take x, y such that $\|x-y\|_\infty \le \frac{\epsilon}{M}$. Then, by using the second triangle inequality we obtain:

$$|\; \|x\| - \|y\| \;| \le \|x - y\| \le M \max \delta_i \le M \frac{\epsilon}{M} = \epsilon.$$

This means that we can make $\|y\|$ arbitrarily close to $\|x\|$ by making y close enough to x in the sense of the defined metric. Therefore $\|.\|$ is continuous. $\square$

Equivalence of norms:

In finite dimensional spaces ($\mathbb{R}^n$, $\mathbb{C}^n$, ..) all norms are 'equivalent': if ϕ_1 and ϕ_2 are two norms then there exists positive constants α, β such that,

$$\beta\phi_2(x) \le \phi_1(x) \le \alpha\phi_2(x)$$

✍7 How can you prove this result? [Hint: Show for $\phi_2 = \|.\|_\infty$]

➤ We can bound one norm in terms of any other norm.

✍8 Show that for any x: $\frac{1}{\sqrt{n}}\|x\|_1 \le \|x\|_2 \le \|x\|_1$

✍9 What are the "unit balls" $B_p = \{x \mid \|x\|_p \le 1\}$ associated with the norms $\|.\|_p$ for $p = 1, 2, \infty$, in $\mathbb{R}^2$?

Convergence of vector sequences

A sequence of vectors $x^{(k)}$, $k = 1, \ldots, \infty$ converges to a vector x with respect to the norm $\|.\|$ if, by definition,

$$\lim_{k \to \infty} \|x^{(k)} - x\| = 0$$

➤ Important point: because all norms in $\mathbb{R}^n$ are equivalent, the convergence of $x^{(k)}$ w.r.t. a given norm implies convergence w.r.t. any other norm.

➤ Notation:

$$\boxed{\lim_{k \to \infty} x^{(k)} = x}$$

Example: The sequence

$$x^{(k)} = \begin{pmatrix} 1 + 1/k \\ \frac{k}{k+\log_2 k} \\ \frac{1}{k} \end{pmatrix}$$

converges to

$$x = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

➤ Note: Convergence of $x^{(k)}$ to x is the same as the convergence of each individual component $x_i^{(k)}$ of $x^{(k)}$ to the corresoponding component x_i of x.

Matrix norms

➤ Can define matrix norms by considering $m \times n$ matrices as vectors in $\mathbb{R}^{mn}$. These norms satisfy the usual properties of vector norms, i.e.,

1. $\|A\| \geq 0, \ \forall \ A \ \in \mathbb{C}^{m\times n}$, and $\|A\| = 0$ iff $A = 0$
2. $\|\alpha A\| = |\alpha| \|A\|, \forall \ A \ \in \mathbb{C}^{m\times n}, \ \forall \ \alpha \in \mathbb{C}$
3. $\|A + B\| \leq \|A\| + \|B\|, \ \forall \ A, B \ \in \mathbb{C}^{m\times n}$.

➤ However, these will lack (in general) the right properties for composition of operators (product of matrices).

➤ The case of $\|.\|_2$ yields the Frobenius norm of matrices.

➤ Given a matrix A in $\mathbb{C}^{m\times n}$, define the set of matrix norms

$$\|A\|_p = \max_{x\in\mathbb{C}^n,\ x\neq 0} \frac{\|Ax\|_p}{\|x\|_p}.$$

➤ These norms satisfy the usual properties of vector norms (see previous page).

➤ The matrix norm $\|.\|_p$ is induced by the vector norm $\|.\|_p$.

➤ Again, important cases are for $p = 1, 2, \infty$.

➤ Show that $\|A\|_p = \max_{x\in\mathbb{C}^n,\ \|x\|_p=1} \|Ax\|_p$

Consistency / sub-mutiplicativity of matrix norms

➤ A fundamental property of matrix norms is consistency

$$\|AB\|_p \le \|A\|_p \|B\|_p.$$

[Also termed "sub-multiplicativity"]

➤ Consequence: (for square matrices) $\boxed{\|A^k\|_p \le \|A\|_p^k}$

➤ A^k converges to zero if any of its p-norms is < 1

[Note: sufficient but not necessary condition]

Frobenius norms of matrices

➤ The Frobenius norm of a matrix is defined by

$$\|A\|_F = \left(\sum_{j=1}^{n} \sum_{i=1}^{m} |a_{ij}|^2\right)^{1/2}.$$

➤ Same as the 2-norm of the column vector in $\mathbb{C}^{mn}$ consisting of all the columns (respectively rows) of A.

➤ This norm is also consistent [but not induced from a vector norm]

✍10 Compute the Frobenius norms of the matrices

$$\begin{pmatrix} \mathbf{1} & \mathbf{1} \\ \mathbf{1} & \mathbf{0} \\ \mathbf{3} & \mathbf{2} \end{pmatrix} \quad \begin{pmatrix} \mathbf{1} & \mathbf{2} & \mathbf{-1} \\ \mathbf{-1} & \sqrt{\mathbf{5}} & \mathbf{0} \\ \mathbf{-1} & \mathbf{1} & \sqrt{\mathbf{2}} \end{pmatrix}$$

✍11 Prove that the Frobenius norm is consistent [Hint: Use Cauchy-Schwartz]

✍12 Define the 'vector 1-norm' of a matrix $\boldsymbol{A}$ as the 1-norm of the vector of stacked columns of $\boldsymbol{A}$. Is this norm a consistent matrix norm?

[*Hint:* Result is true – Use Cauchy-Schwarz to prove it.]

Expressions of standard matrix norms

➤ Recall the notation: (for square $n \times n$ matrices)

$\rho(A) = \max |\lambda_i(A)|$; $Tr(A) = \sum_{i=1}^{n} a_{ii} = \sum_{i=1}^{n} \lambda_i(A)$

where $\lambda_i(A)$, $i = 1, 2, \ldots, n$ are all eigenvalues of A

$$\|A\|_1 = \max_{j=1,\ldots,n} \sum_{i=1}^{m} |a_{ij}|,$$

$$\|A\|_\infty = \max_{i=1,\ldots,m} \sum_{j=1}^{n} |a_{ij}|,$$

$$\|A\|_2 = \left[\rho(A^H A)\right]^{1/2} = \left[\rho(AA^H)\right]^{1/2},$$

$$\|A\|_F = \left[Tr(A^H A)\right]^{1/2} = \left[Tr(AA^H)\right]^{1/2}.$$

✍13 Compute the p-norm for $p = 1, 2, \infty, F$ for the matrix

$$A = \begin{pmatrix} 0 & 2 \\ 0 & 1 \end{pmatrix}$$

✍14 Show that $\rho(A) \leq \|A\|$ for any matrix norm.

✍15 Is $\rho(A)$ a norm?

1. $\rho(A) = \|A\|_2$ when A is Hermitian ($A^H = A$). ➤ True for this particular case...

2. ... However, not true in general. For

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},$$

we have $\rho(A) = 0$ while $A \neq 0$. Also, triangle inequality not satisfied for the pair A, and $B = A^T$. Indeed, $\rho(A + B) = 1$ while $\rho(A) + \rho(B) = 0$.

Singular values and matrix norms

- Let $A \in \mathbb{R}^{m\times n}$ or $A \in \mathbb{C}^{m\times n}$
- Eigenvalues of $A^H A$ & AA^H are real ≥ 0. ✍16 Show this.
- Let $\begin{cases} \sigma_i = \sqrt{\lambda_i(A^H A)} \ i = 1, \cdots, n & \text{if } n \leq m \\ \sigma_i = \sqrt{\lambda_i(AA^H)} \ i = 1, \cdots, m & \text{if } m < n \end{cases}$
- The σ_i's are called singular values of A.
- Note: a total of $\min(m, n)$ singular values.
- Always sorted decreasingly: $\sigma_1 \geq \sigma_2 \geq \sigma_3 \geq \cdots \sigma_k \geq \cdots$
- We will see a lot more on singular values later

- Assume we have r nonzero singular values:

$$\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_r > 0$$

- Then:
  - $\|A\|_2 = \sigma_1$
  - $\|A\|_F = \left[\sum_{i=1}^r \sigma_i^2\right]^{1/2}$

- More generally: Schatten p-norm ($p \geq 1$) defined by

$$\|A\|_{*,p} = \left[\sum_{i=1}^r \sigma_i^p\right]^{1/p}$$

- Note: $\|A\|_{*,p} = p$-norm of vector $[\sigma_1; \sigma_2; \cdots ; \sigma_r]$

- In particular: $\|A\|_{*,1} = \sum \sigma_i$ is called the nuclear norm and is denoted by $\|A\|_*$. (Common in machine learning).

A few properties of the 2-norm and the F-norm

➤ Let $A = uv^T$. Then $\|A\|_2 = \|u\|_2 \|v\|_2$

✍17 Prove this result

✍18 In this case $\|A\|_F = ??$

For any $A \in \mathbb{C}^{m\times n}$ and unitary matrix $Q \in \mathbb{C}^{m\times m}$ we have

$$\|QA\|_2 = \|A\|_2; \quad \|QA\|_F = \|A\|_F.$$

✍19 Show that the result is true for any orthogonal matrix Q (Q has orthonomal columns), i.e., when $Q \in \mathbb{C}^{p\times m}$ with $p > m$

✍20 Let $Q \in \mathbb{C}^{n\times n}$. Do we have $\|AQ\|_2 = \|A\|_2$? $\|AQ\|_F = \|A\|_F$? What if $Q \in \mathbb{C}^{n\times p}$, with $p < n$?