✍ 1 *Solution of System* $\begin{pmatrix} 5 & 10 & 25 \\ 1 & 1 & 1 \\ 0 & 10 & 25 \end{pmatrix} \begin{pmatrix} x_n \\ x_d \\ x_q \end{pmatrix} = \begin{pmatrix} 145 \\ 12 \\ 125 \end{pmatrix}$

Solution: *You will find:* $x_n = 4, \quad x_d = 5, \quad x_q = 3.$

✍ 3 $(A^T)^T = ??$ **Solution:** $(A^T)^T = A$

✍ 4 $(AB)^T = ??$ **Solution:** $(AB)^T = B^T A^T$

✍ 5 $(A^H)^H = ??$ **Solution:** $(A^H)^H = A$

✍ 6 $(A^H)^T = ??$ **Solution:** $(A^H)^T = \bar{A}$

✍ 7 $(ABC)^T = ??$ **Solution:** $(ABC)^T = C^T B^T A^T$

✍ 8 *True/False:* $(AB)C = A(BC)$ **Solution:** $\rightarrow$ *True*

✍ 9 *True/False:* $AB = BA$ **Solution:** $\longrightarrow$ *false*

✍ 10 *True/False:* $AA^T = A^T A$ **Solution:** $\longrightarrow$ *false in general*

✍ 12 *Complexity? [number of multiplications and additions for matrix multiply]*

Solution: *Let* $A \in \mathbb{R}^{m\times n}$ *and* $B \in \mathbb{R}^{n\times p}$. *Then the product* AB *requires* $2mnp$ *operations (there are* mp *entries in all and each of them requires* $2n$ *operations).* □

✍ 13 *What happens to these 3 different approches to matrix-matrix multiplication when* B *has one column* $(p = 1)$*?*

Solution: *In the first:* $C_{:,j}$ *the* j*=th column of* C *is a linear combination of the columns of* A*. This is the usual matrix-vector product.*

In the second: $C_{i,:}$ *is just a number which is the inner product of the* i*th row of* A *with the column* B*.*

The 3rd formula will give the exact same expression as the first. □

✍ 14 *Characterize the matrices* AA^T *and* $A^T A$ *when* A *is of dimension* $n \times 1$*.*

Solution: *When* $A \in \mathbb{R}^{n\times 1}$ *then* AA^T *is a rank-one* $n \times n$ *matrix and* $A^T A$ *is a scalar: the inner product of the column* A *with itself.* □

✍15 ✍16 *Show that* $A \in \mathbb{R}^{m\times n}$ *is of rank one iff [if and only if] there exist two nonzero vectors* $u \in \mathbb{R}^m$ *and* $v \in \mathbb{R}^n$ *such that*

$$A = uv^T.$$

What are the eigenvalues and eigenvectors of A*?*

Solution: *(a)*

$\leftarrow$ *First we show that: When both* u *and* v *are nonzero vectors then the rank of a matrix of the matrix* $A = uv^T$ *is one. The range of* A *is the set of all vectors of the form*

$$y = Ax = uv^T x = (v^T x)u$$

since u *is a nonzero vector, and not all vectors* $v^T x$ *are zero (because* $v \neq 0$*) then this space is* *of dimension 1.*

$\rightarrow$ *Next we show that: If* A *is of rank one than there exist nonzero vectors* u, v *such that* $A = uv^T$. *If* A *is of rank one, then* $Ran(A) = Span\{u\}$ *for some nonzero vector* u. *So for every vector* x, *the vector* Ax *is a multiple of* u. *Let* $e_1, e_2, \cdots, e_n$ *the vectors of the canonical basis of* $\mathbb{R}^n$ *and let* $\nu_1, \nu_2, \cdots, \nu_n$ *the scalars such that* $Ae_i = \nu_i u$. *Define* $v = [\nu_1, \nu_2, \cdots, \nu_n]^T$. *Then* $A = uv^T$ *because the matrices* A *and* uv^T *have the same columns. (Note that the* j*-th column of* A *is the vector* Ae_j*). In addition,* $v \neq 0$ *otherwise* $A == 0$ *which would be a contradiction because rank(*A*) = 1.*

(b) Eigenvalues /vectors

Write $Ax = \lambda x$ *then notice that this means* $(v^T x)u = \lambda x$ *so either* $v^T x = 0$ *and* $\lambda = 0$ *or* $x = u$ *and* $\lambda = v^T u$. *Two eigenvalues: 0 and* $v^T x$... $\square$

✍ 17 *Is it true that*

$$\mathrm{rank}(A) = \mathrm{rank}(\bar{A}) = \mathrm{rank}(A^T) = \mathrm{rank}(A^H) \ ?$$

Solution:

The answer is yes and it follows from the fact that the ranks of A *and* A^T *are the same and the ranks of* A *and* $\bar{A}$ *are also the same.*

It is known that $rank(A) = rank(A^T)$. *We now compare the ranks of* A *and* $\bar{A}$ *(everything is considered to be complex).*

The important property that is used is that if a set of vectors is linearly independent then so is its conjugate. [convince yourself of this by looking at material from 2033]. If A *has rank* r *and for example its first* r *columns are the basis of the range, the the same* r *columns of* $\bar{A}$ *are also linearly independent. So* $rank(\bar{A}) \geq rank(A)$. *Now you can use a similar argument to show that* $rank(A) \geq rank(\bar{A})$. *Therefore the ranks are the same.* $\square$

✍ 21 *Eigenvalues of two similar matrices A and B are the same. What about eigenvectors?*

Solution: *If $Au = \lambda u$ then $XBX^{-1}u = \lambda u \rightarrow B(X^{-1}u) = \lambda(X^{-1}u) \rightarrow \lambda$ is an eigenvalue of B with eigenvector $X^{-1}u$ (note the vector $X^{-1}u$ cannot be equal to zero because $u \neq 0$.)* ☐

✍ 22 *Given a polynomial $p(t)$ how would you define $p(A)$?*

Solution: *If $p(t) = \alpha_0 + \alpha_1 t + \alpha_2 t^2 + \cdots + \alpha_k t^k$ then*

$$p(A) = \alpha_0 I + \alpha_1 A + \alpha_2 A^2 + \cdots + \alpha_k A^k$$

where

$$A^j = \underbrace{A \times A \times \cdots \times A}_{j \text{ times}}$$

☐

✍ 23 *Given a function $f(t)$ (e.g., e^t) how would you define $f(A)$? [You may limit yourself to the case when A is diagonalizable]*

Solution: *The easiest way would be through the Taylor series expansion..*

$$f(A) = f(0)I + \frac{f'(0)}{1!}A + \frac{f''(0)}{2!}A^2 \cdots \frac{f^{(k)}(0)}{k!}A^k + \cdots$$

However, this will require a justification: Will this expression 'converge' as the number of terms goes to infinity? This is where norms are useful. We will revisit this in next set. $\square$

✍ 24 *If A is nonsingular what are the eigenvalues/eigenvectors of A^{-1}?*

Solution: *Assume that $Au = \lambda u$. Multiply both sides by the inverse of A: $u = \lambda A^{-1}u$ - then by the inverse of λ: $\lambda^{-1}u = A^{-1}u$. Therefore, $1/\lambda$ is an eigenvalue and u is an associated eigenvector.* $\square$

✍ 25 *What are the eigenvalues/eigenvectors of A^k for a given integer power k?*

Solution: *Assume that $Au = \lambda u$. Multiply both sides by A and repeate k times. You will get $A^k u = \lambda^k u$. Therefore, λ^k is an eigenvalue of A^k and u is an associated eigenvector.* $\square$

✍ 26 *What are the eigenvalues/eigenvectors of $p(A)$ for a polynomial p?*

Solution: *Using the previous result you can show that $p(\lambda)$ is an eigenvalue of $p(A)$ and u is an associated eigenvector.* ☐

✍27 *What are the eigenvalues/eigenvectors of $f(A)$ for a function f? [Diagonalizable case]*

Solution: *This will require using the diagonalized form of A: $A = XDX^{-1}$. With this $f(A) = Xf(D)X^{-1}$. It becomes clear that the eigenvalues are the diagonal entries of $f(D)$, i.e., the values $f(\lambda_i)$ for $i = 1, \cdots, n$. As for the eigenvectors - recall that they are the columns of the X matrix in the diagonalized form – And X is the same for A and $f(A)$. So the eigenvectors are the same.* ☐

✍28 *For two $n \times n$ matrices A and B are the eigenvalues of AB and BA the same?*

Solution: *We will show that if λ is an eigenvalue of AB then it is also an eigenvalue of BA. Assume that $ABu = \lambda u$ and multiply both sides by B. Then $BABu = \lambda Bu$ – which we write in the form: $BAv = \lambda v$ where $v = Bu$. In the situation when $v \neq 0$, we clearly see that λ is a nonzero eigenvalue of BA with the associated eigenvector v. We now deal with the case when $v = 0$. In this case, since $ABu = \lambda u$, and $u \neq 0$ we must have $\lambda = 0$. However,* *clearly* $\lambda = 0$ *is also an eigenvalue of* BA *because* $\det(BA) = det(AB) = 0$.

We can similarly show that any eigenvalue of BA *are also eigenvalues of* AB *by interchanging the roles of* A *and* B. *This completes the proof* □

✍ 30 *Trace, spectral radius, and determinant of* $A = \begin{pmatrix} 2 & 1 \\ 3 & 0 \end{pmatrix}$.

Solution: *Trace is 2, determinant is* -3. *Eigenvalues are* $3, -1$ *so* $\rho(A) = 3$. □

✍ 31 *What is the inverse of a unitary (complex) or orthogonal (real) matrix?*

Solution: *If* Q *is unitary then* $Q^{-1} = Q^H$. □

✍ 32 *What can you say about the diagonal entries of a skew-symmetric (real) matrix?*

Solution: *They must be equal to zero.* □

✍ 33 *What can you say about the diagonal entries of a Hermitian (complex) matrix?*

Solution: *We must have* $a_{ii} = \bar{a}_{ii}$. *Therefore* a_{ii} *must be real.* □

✍ 34 *What can you say about the diagonal entries of a skew-Hermitian (complex) matrix?*

Solution: *We must have* $a_{ii} = -\bar{a}_{ii}$*. Therefore* a_{ii} *must be purely imaginary.* ☐

✍ 35 *Which matrices of the following type are also normal: real symmetric, real skew-symmetric, Hermitian, skew-Hermitian, complex symmetric, complex skew-symmetric matrices.*

Solution: *Real symmetric, real skew-symmetric, Hermitian, skew-Hermitian matrices are normal. Complex symmetric, complex skew-symmetric matrices are not necessarily normal.* ☐

✍ 39 *What does the matrix-vector product* Va *represent?*

Solution: *If* $a = [a_0, a_2, \cdots, a_n]$ *and* $p(t)$ *is the* n*-th degree polymomial:*

$$p(t) = a_0 + a_1 t + a_2 t^2 + \cdots a_n t^n$$

then Va *is a vector whose components are the values* $p(x_0), p(x_1), \cdots, p(x_n)$*.* ☐

✍ 40 *Interpret the solution of the linear system* $Va = y$ *where* a *is the unknown. Sketch a 'fast' solution method based on this.*

Solution: *Given the previous exercise, the interpretation is that we are seeking a polynomial of degree* n *whose values at* $x_0, \cdots, x_n$ *are the components of the vector* y*, i.e.,* $y_0, y_1, \cdots, y_n$*. This is known as polynomial interpolation (see csci 5302). The polynomial can be determined by, e.g., the Newton table in* $O(n^2)$ *operations.* $\square$